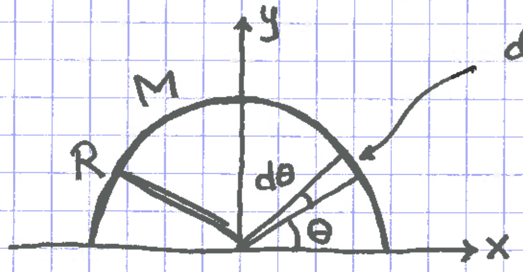
Eks 1: CM for halvsirkulær bøyle



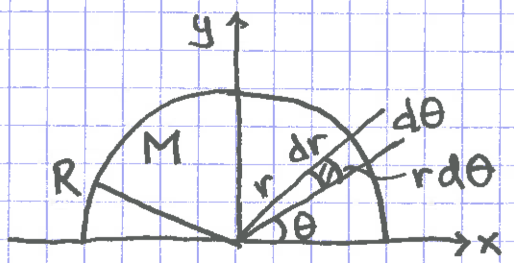
$$dm = M \cdot \frac{ds}{\pi R} = M \frac{R\,d\theta}{\pi R} = M \frac{d\theta}{\pi}$$

$$\vec{r} = \hat{x}x + \hat{y}y = \hat{x}R\cos\theta + \hat{y}R\sin\theta$$

$$\vec{R}_{CM} = X_{CM}\hat{x} + Y_{CM}\hat{y} \ ; \quad X_{CM} = 0 \text{ pga symmetri}$$

$$Y_{CM} = \frac{1}{M}\int y\,dm = \frac{R}{\pi}\int_0^\pi \sin\theta\,d\theta = \underline{\underline{\frac{2}{\pi}R \approx 0.64R}}$$

Eks 2: CM for halvsirkulær tynn skive



$$\frac{dm}{M} = \frac{r\,d\theta \cdot dr}{\frac{1}{2}\pi R^2}$$

$$X_{CM} = 0 \quad \text{pga symmetri}$$

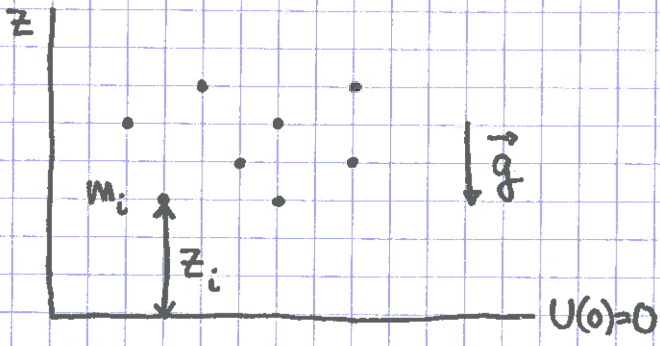
$$Y_{CM} = \int y\,\frac{dm}{M} = \frac{2}{\pi R^2}\int_{r=0}^{R}\int_{\theta=0}^{\pi} r\sin\theta \cdot r\,d\theta \cdot dr$$

$$= \frac{2}{\pi R^2}\underbrace{\left\{\int_0^R r^2\,dr\right\}}_{=R^3/3} \cdot \underbrace{\left\{\int_0^\pi \sin\theta\,d\theta\right\}}_{=\left.(-\cos\theta)\right|_0^\pi = 2} = \underline{\underline{\frac{4}{3\pi}R \approx 0.42R}}$$

Eks 3: CM for kompakt halvkule

$$Y_{CM} = \frac{3}{8}R \qquad \text{(regn ut selv)}$$

Potensiell energi for partikkelsystem i tyngdefeltet



$$U = \sum_i U_i = \sum_i m_i g z_i$$

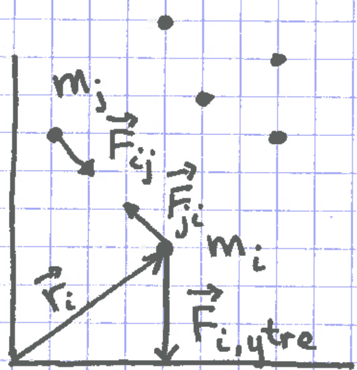
Hvis g er konstant:

$$U = g \sum_i m_i z_i = \underline{g M Z_{CM}}$$

Dvs: Som om hele massen M er samlet i høyden

$$Z_{CM} = \frac{1}{M} \sum_i m_i z_i \qquad (\text{f.eks. i } \vec{R}_{CM})$$

Tyngdepunktbevegelsen [YF 8.5 ; LL 5.8]



System med N punktmasser $m_1, m_2, \dots, m_N$.

N2 for m_i:

$$m_i \ddot{\vec{r}}_i = \underbrace{\vec F_{i,ytre}}_{\text{netto ytre kraft på } m_i} + \underbrace{\sum_{j \neq i} \vec F_{ji}}_{\text{netto indre kraft på } m_i} \qquad (i = 1, 2, \dots, N)$$

Legger sammen N2 for alle massene:

$$\sum_i m_i \ddot{\vec{r}}_i = \underbrace{\sum_i \vec F_{i,ytre}}_{\text{netto ytre kraft } \vec F_{ytre} \text{ på hele systemet}} + \underbrace{\sum_i \sum_{j \neq i} \vec F_{ji}}_{= \vec F_{21} + \vec F_{12} + \dots + \vec F_{N,N-1} + \vec F_{N-1,N} = 0 \text{ pga N3}}$$

$$\sum_i m_i \ddot{\vec{r}}_i = \frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \frac{d^2}{dt^2}\left\{ M \vec{R}_{CM} \right\} = M \ddot{\vec{R}}_{CM}$$

$$\Rightarrow \quad \boxed{M \ddot{\vec{R}}_{CM} = \vec{F}_{ytre}}$$

Dvs: CM beveger seg som om hele massen M er samlet i $\vec{R}_{CM}$ og blir utsatt for summen av alle ytre krefter som virker på systemet.

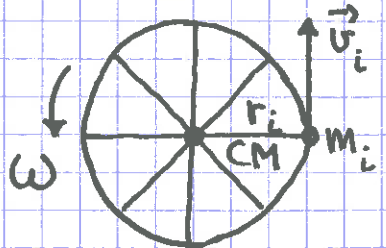
I tillegg til translasjon av CM:

- Rotasjon om CM
- Vibrasjon om CM

ROTASJON [YF 9, 10 ; LL 6 (5)]

Innledning.

- Roterende hjul



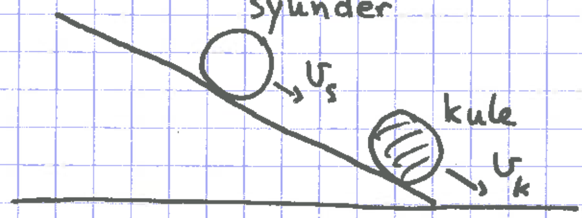
$K_{trans} = 0$ da CM er i ro

$K_{rot} \neq 0$

$\vec{P} = \sum_i m_i \vec{v}_i = 0$ (lineær impuls)

$\vec{L} \neq 0$ (dreieimpuls)

- Rulling

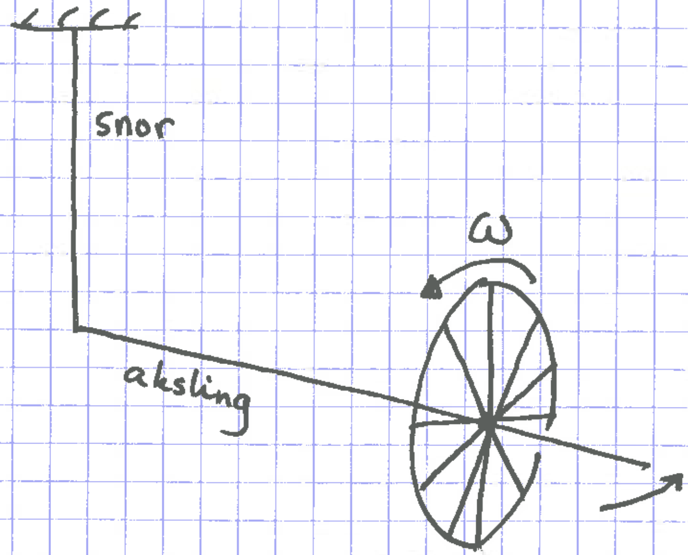


Kreftenes angrepspunkt; dreiemoment.

Hvorfor $v_k > v_s$?

Friksjon.

- Overraskende dynamikk



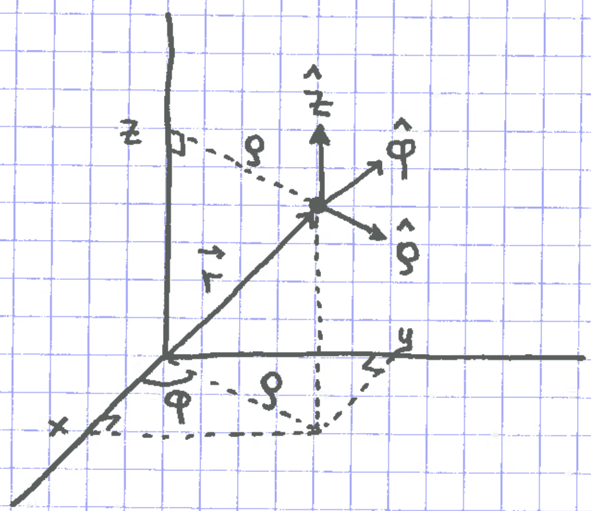
Presesjon
Gyroskop

- Stive legemer : Alle masseelementer i <u>fast</u> innbyrdes avstand. Kun translasjon og rotasjon ; ikke vibrasjon.

Sirkelbevegelse [YF 9.1–9.3 ; LL 1,8]

Anta rotasjon om gitt akse, f.eks. z-aksen

Lurt med sylinderkoord. = "polarkoord. + z"



$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = \varrho\hat{\varrho} + z\hat{z}$$

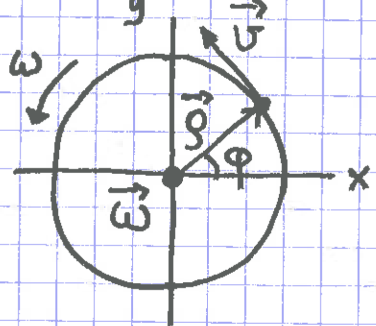
$$x = \varrho\cos\varphi, \quad y = \varrho\sin\varphi, \quad z = z$$

$$\varrho = \sqrt{x^2+y^2}, \quad \tan\varphi = y/x$$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{\varrho^2+z^2}$$

Rotasjonsaksen pekes ut ved hjelp av ω:

$$\omega \longrightarrow \vec{\omega} = \omega\hat{z}$$



$$\vec{v} = \frac{d\vec{s}}{dt} = \varrho\frac{d\varphi}{dt}\hat{\varphi} = \varrho\omega\hat{\varphi}$$

$$\vec{\omega} = \omega\hat{z}, \quad \vec{\varrho} = \varrho\hat{\varrho}$$

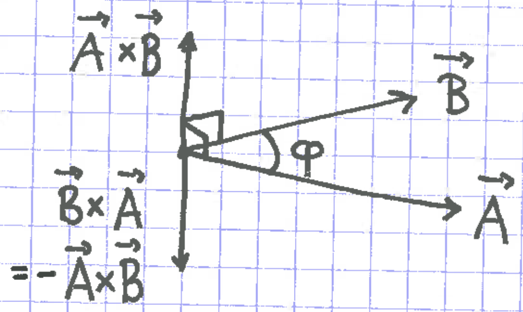
[Notasjon for vektorer ⊥ planet:

• opp   × ned]

Med kryssprodukt: $\boxed{\vec{v} = \vec{\omega}\times\vec{\varrho}}$

Retning: $\hat{\varphi} = \hat{z}\times\hat{\varrho}$, OK! Abs.verdi: $v = \omega\varrho$, OK!

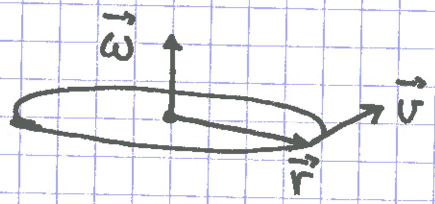
Kryssprodukt (Vektorprodukt):



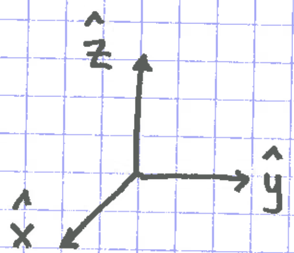
$$|\vec{A}\times\vec{B}| = A\cdot B\cdot\sin\varphi$$

Høyrehåndsregel gir retningen.

For sirkelbevegelsen:

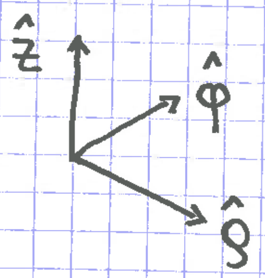


For enhetsvektorer:



$\hat{x}\times\hat{y} = \hat{z}$, $\hat{y}\times\hat{z} = \hat{x}$, $\hat{z}\times\hat{x} = \hat{y}$

$\hat{y}\times\hat{x} = -\hat{z}$ osv.

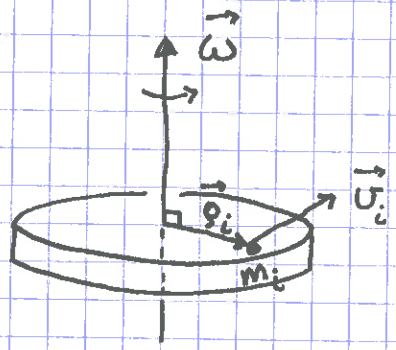


$\hat{\rho}\times\hat{\varphi} = \hat{z}$, $\hat{\varphi}\times\hat{z} = \hat{\rho}$,

$\hat{z}\times\hat{\rho} = \hat{\varphi}$

$$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \ldots = 0$$

Rotasjonsenergi [YF 9.4 ; LL 6.4]



$$\vec{v}_i = \vec{\omega} \times \vec{\rho}_i$$

$$v_i = \omega \rho_i$$

$$K = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (\rho_i \omega)^2 = \frac{1}{2} \left\{ \sum_i m_i \rho_i^2 \right\} \omega^2$$

Treghetsmoment [YF 9.4 ; LL 6.3]

$I = \sum_i m_i \rho_i^2$ = legemets treghetsmoment mhp gitt akse

Med kontinuerlig massefordeling:

$m_i \rightarrow \Delta m_i \xrightarrow{\Delta m_i \to 0} dm$; $\sum_i \rightarrow \int$ over legemet

$$\Rightarrow \quad \boxed{I = \int \rho^2 \, dm}$$

ρ = avstand fra aksen til dm

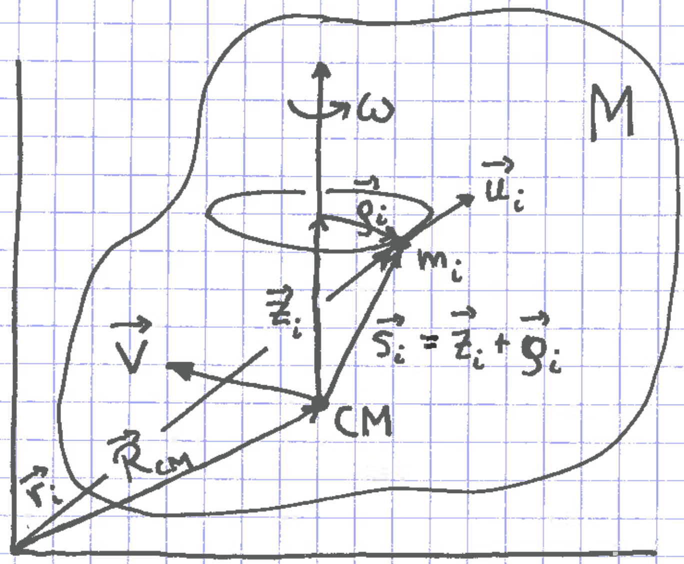
Dermed:

$$\boxed{K_{rot} = \frac{1}{2} I \omega^2}$$

Kin. energi for stivt legeme [YF 10.3 ; LL 6.6]

Generell bevegelse for stivt legeme:

Translasjon av CM + Rotasjon om akse gjennom CM



$$\vec{r}_i = \vec{R}_{CM} + \vec{s}_i$$

$$\vec{v}_i = \dot{\vec{r}}_i = \dot{\vec{R}}_{CM} + \dot{\vec{s}}_i = \vec{V} + \vec{u}_i$$

$$v_i^2 = V^2 + u_i^2 + 2\vec{V}\cdot\vec{u}_i$$

$$u_i^2 = (\varrho_i \omega)^2$$

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i V^2 + \frac{1}{2}\sum_i m_i u_i^2 + \vec{V}\cdot\sum_i m_i \vec{u}_i$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}\underbrace{\left(\sum_i m_i \varrho_i^2\right)}_{=I_0}\omega^2 + \vec{V}\cdot\frac{d}{dt}\sum_i m_i \vec{s}_i$$

$$\sum_i m_i \vec{s}_i = \sum_i m_i(\vec{r}_i - \vec{R}_{CM}) = M\vec{R}_{CM} - M\vec{R}_{CM} = 0$$

$$\Rightarrow \boxed{K = K_{trans} + K_{rot} = \frac{1}{2}MV^2 + \frac{1}{2}I_0\omega^2}$$

M = legemets masse

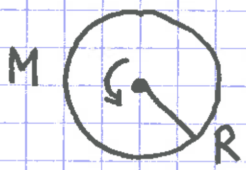
$V = \dot{\vec{R}}_{CM}$ = hastigheten til CM

I_0 = legemets tregh.mom. mhp rot.aksen gjennom CM

$\vec{\omega}$ = vinkelhastigheten for rot. om —— " ——

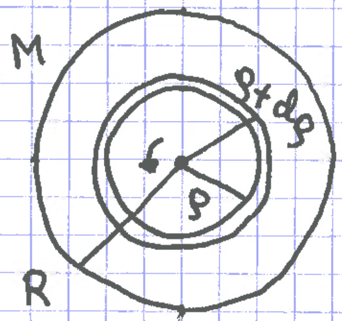
Beregning av I [YF 9.6 ; LL 6.3]

Eks 1: Ring, Sylinderskall



$$I_0 = \int \rho^2 \, dm = R^2 \int dm = \underline{MR^2}$$

Eks 2: Rund skive, Kompakt sylinder

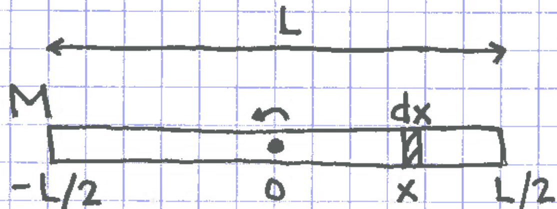


$$\frac{dm}{M} = \frac{dA}{A} = \frac{2\pi \rho \, d\rho}{\pi R^2}$$

$$I_0 = \int_0^R \rho^2 M \frac{2\pi \rho \, d\rho}{\pi R^2} = \frac{2M}{R^2} \Big|_0^R \frac{1}{4}\rho^4$$

$$= \underline{\frac{1}{2} MR^2}$$

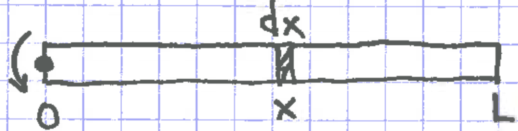
Eks 3: Tynn stang



$$\frac{dm}{M} = \frac{dx}{L}$$

$$I_0 = \int_{-L/2}^{L/2} x^2 M \frac{dx}{L} = \frac{M}{L} \Big|_{-L/2}^{L/2} \frac{1}{3} x^3 = \underline{\frac{1}{12} ML^2}$$

Eks 4: Akse gjennom stangas ende



$$I_0 = \int_0^L x^2 M \frac{dx}{L} = \underline{\frac{1}{3} ML^2}$$

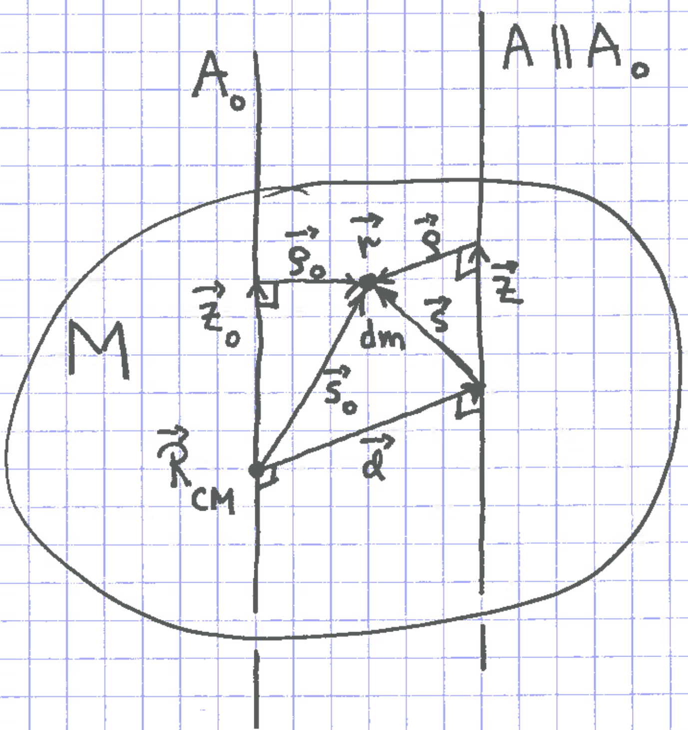
Eks 5: Kuleskall $I_0 = \frac{2}{3} MR^2$

Eks 6: Kompakt kule $I_0 = \frac{2}{5} MR^2$

} Øving.

Steiners sats

(Parallellakseteoremet) [YF 9.5; LL 6.3]



Fra figuren:

$$\vec{r} = \vec{R}_{CM} + \vec{s}_0$$

$$\vec{s}_0 = \vec{z}_0 + \vec{\rho}_0$$

$$\vec{s} = \vec{z} + \vec{\rho}$$

$$\vec{z} = \vec{z}_0 \qquad (\vec{d} \perp \hat{z})$$

$$\Rightarrow \vec{\rho} - \vec{\rho}_0 = \vec{s} - \vec{s}_0 = -\vec{d}$$

$$\Rightarrow \rho^2 = (\vec{\rho}_0 - \vec{d}) \cdot (\vec{\rho}_0 - \vec{d})$$

$$= \rho_0^2 + d^2 - 2\vec{d} \cdot \vec{\rho}_0$$

$$I = \int \rho^2 dm = \underbrace{\int \rho_0^2 dm}_{I_0} + d^2 \cdot \underbrace{\int dm}_{M} - 2 \int \vec{d} \cdot \vec{\rho}_0 \, dm$$

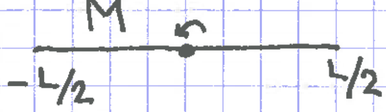
$$\vec{d} \cdot \vec{\rho}_0 = \vec{d} \cdot (\vec{s}_0 - \vec{z}_0) = \vec{d} \cdot \vec{s}_0 = \vec{d} \cdot (\vec{r} - \vec{R}_{CM})$$

$$\Rightarrow \int \vec{d} \cdot \vec{\rho}_0 \, dm = \vec{d} \cdot \underbrace{\int \vec{r} \, dm}_{M\vec{R}_{CM}} - \vec{d} \cdot \vec{R}_{CM} \underbrace{\int dm}_{M} = 0$$

$$\Rightarrow \boxed{I = I_0 + Md^2}$$

Eks 1: Tynn stang

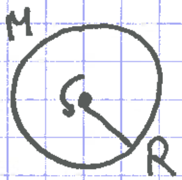
$(d = L/2)$



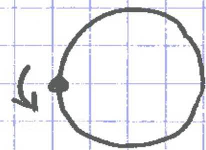
$\Rightarrow$

$$I_0 = \frac{1}{12} ML^2 \qquad I = I_0 + M\left(L/2\right)^2 = \frac{1}{3} ML^2$$

Eks 2: Kuleskall



$$I_0 = \frac{2}{3} MR^2 \quad \Rightarrow$$



$d = R$

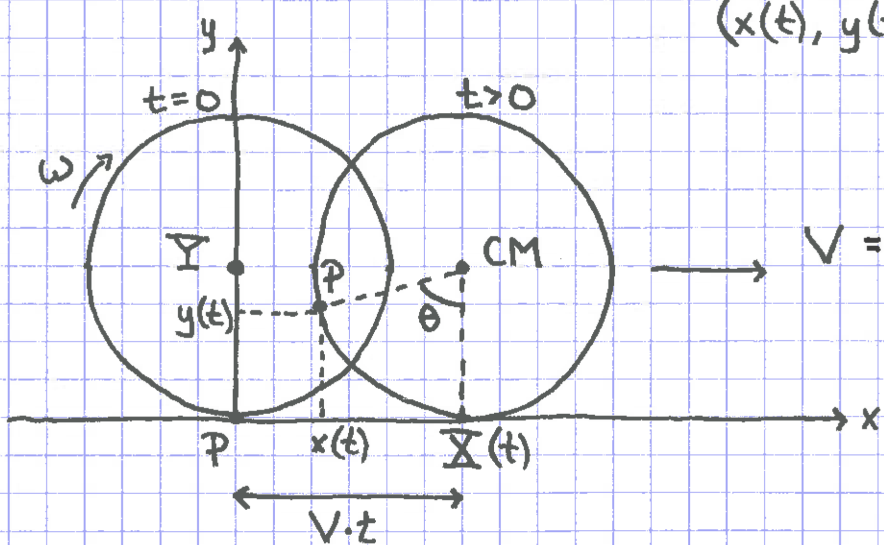
$$I = I_0 + MR^2 = \frac{5}{3} MR^2$$

Rulling og skuring [YF 10.3 ; LL 6.7]

Ren rulling

P = punkt på periferien

$(x(t), y(t))$ = banen til P



$$\longrightarrow \quad V = \dot{R}_{CM} = \dot{X} \quad ; \quad Y = R$$

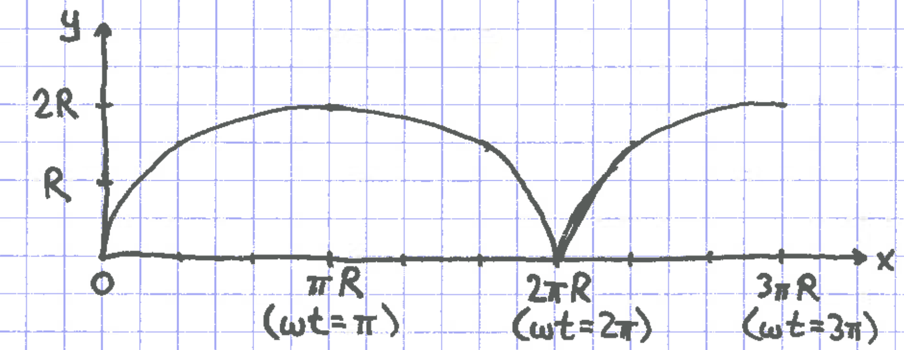
$$\omega = \frac{d\theta}{dt} \quad , \quad X = Vt = R\theta$$

$$V = \dot{X} = R\dot{\theta} = R\omega \, , \quad A = \ddot{X} = R\ddot{\theta} = R\dot{\omega} = R\alpha$$

Rullebetingelser: $\boxed{V = R\omega \, , \quad A = R\alpha}$

Banen til P (anta ω = konst. $\Rightarrow \theta = \omega t$):

$$x(t) = X(t) - R\sin\theta = Vt - R\sin\omega t$$
$$y(t) = Y - R\cos\theta = R - R\cos\omega t$$



Sykloide

Hastigheten til P:

$$\dot{x} = V - \omega R\cos\omega t = V(1-\cos\omega t)$$
$$\dot{y} = \omega R\sin\omega t = V\sin\omega t$$

$$\Rightarrow \quad \vec{v}(\theta) = \hat{x}\, V(1-\cos\theta) + \hat{y}\, V\sin\theta$$