CHAPTER

22

The Electric Field II: Continuous Charge Distributions

CO22 art to come

BY DESCRIBING CHARGE IN TERMS OF CONTINUOUS CHARGE DENSITY, IT BECOMES POSSIBLE TO CALCULATE THE CHARGE ON THE SURFACE OF OBJECTS AS LARGE AS CELESTIAL BODIES.

? How would you calculate the charge on the surface of the Earth? (See Example 22-10.)

On a microscopic scale, electric charge is quantized. However, there are often situations in which many charges are so close together that they can be thought of as continuously distributed. The use of a continuous charge density to describe a large number of discrete charges is similar to the use of a continuous mass density to describe air, which actually consists of a large number of discrete molecules. In both cases, it is usually easy to find a volume element ΔV that is large enough to contain a multitude of individual charges or molecules and yet is small enough that replacing ΔV with a differential dV and using calculus introduces negligible error.

We describe the charge per unit volume by the **volume charge density** ρ:

$$\rho = \frac{\Delta Q}{\Delta V} \qquad \text{22-1}$$

Often charge is distributed in a very thin layer on the surface of an object. We define the **surface charge density** σ as the charge per unit area:

$$\sigma = \frac{\Delta Q}{\Delta A} \qquad \text{22-2}$$

Similarly, we sometimes encounter charge distributed along a line in space. We define the **linear charge density** λ as the charge per unit length:

$$\lambda = \frac{\Delta Q}{\Delta L} \qquad 22\text{-}3$$

➢ **In this chapter, we show how Coulomb's law is used to calculate the electric field produced by various types of continuous charge distributions. We then introduce Gauss's law, which relates the electric field on a closed surface to the net charge within the surface, and we use this relation to calculate the electric field for symmetric charge distributions.**

22-1 Calculating $\vec{E}$ From Coulomb's Law

Figure 22-1 shows an element of charge $dq = \rho\, dV$ that is small enough to be considered a point charge. Coulomb's law gives the electric field $d\vec{E}$ at a field point P due to this element of charge as:

$$d\vec{E} = \frac{k\, dq}{r^2}\hat{r}$$

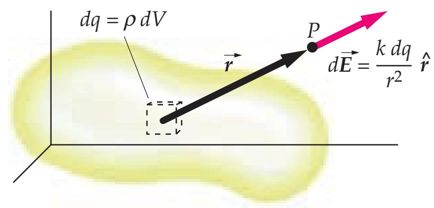


FIGURE 22-1 An element of charge dq produces a field $d\vec{E} = (k\, dq/r^2)\,\hat{r}$ at point P. The field at P is found by integrating over the entire charge distribution.

where $\hat{r}$ is a unit vector that points from the source point to the field point P. The total field at P is found by integrating this expression over the entire charge distribution. That is,

$$\vec{E} = \int_V \frac{k\, dq}{r^2}\hat{r} \qquad 22\text{-}4$$

ELECTRIC FIELD DUE TO A CONTINUOUS CHARGE DISTRIBUTION

where $dq = \rho\, dV$. If the charge is distributed on a surface or line, we use $dq = \sigma\, dA$ or $dq = \lambda\, dL$ and integrate over the surface or line.

$\vec{E}$ on the Axis of a Finite Line Charge

A charge Q is uniformly distributed along the x axis from $x = -\frac{1}{2}L$ to $x = +\frac{1}{2}L$, as shown in Figure 22-2. The linear charge density for this charge is $\lambda = Q/L$. We wish to find the electric field produced by this line charge at some field point P on the x axis at $x = x_P$, where $x_P > \frac{1}{2}L$. In the figure, we have chosen the element of charge dq to be the charge on a small element of length dx at position x. Point P is a distance $r = x_P - x$ from dx. Coulomb's law gives the electric field at P due to the charge dq on this length dx. It is directed along the x axis and is given by

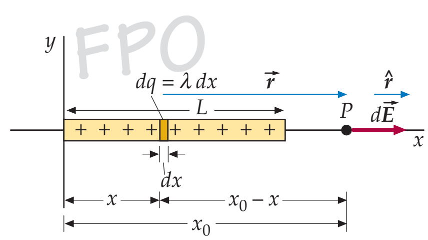


FIGURE 22-2 Geometry for the calculation of the electric field on the axis of a uniform line charge of length L, charge Q, and linear charge density $\lambda = Q/L$. An element $dq = \lambda\, dx$ is treated as a point charge.

$$dE_x\hat{i} = \frac{k\, dq}{(x_P - x)^2}\hat{i} = \frac{k\lambda\, dx}{(x_P - x)^2}\hat{i}$$

We find the total field $\vec{E}$ by integrating over the entire line charge in the direction of increasing x (from $x = -\frac{1}{2}L$ to $x = +\frac{1}{2}L$):

$$E_x = k\lambda \int_{-L/2}^{+L/2} \frac{dx}{(x_P - x)^2} = -k\lambda \int_{x_P-(L/2)}^{x_P+(L/2)} \frac{du}{u^2}$$

where $u = x_P - x$ (so $du = -dx$). Note that if $x = -\frac{1}{2}L$, $u = x_P - \frac{1}{2}L$, and if $x = +\frac{1}{2}L$, $u = x_P + \frac{1}{2}L$. Evaluating the integral gives

$$E_x = -k\lambda \frac{1}{u}\Bigg|_{x_P-(L/2)}^{x_P+(L/2)} = k\lambda\left\{\frac{1}{x_P - \frac{1}{2}L} - \frac{1}{x_P + \frac{1}{2}L}\right\} = \frac{k\lambda L}{x_P^2 - (\frac{1}{2}L)^2}$$

Substituting Q/L for λ, we obtain

$$E_x = \frac{kQ}{x_P^2 - (\frac{1}{2}L)^2}, \qquad x_P > \tfrac{1}{2}L \qquad \text{22-5}$$

We can see that if x_P is much larger than L, the electric field at x_P is approximately kQ/x_P^2. That is, if we are sufficiently far away from the line charge, it approaches that of a point charge Q at the origin.

EXERCISE The validity of Equation 22-5 is established for the region $x_P > \frac{1}{2}L$. Is it also valid in the region $-\frac{1}{2}L \le x_P \le \frac{1}{2}L$? Explain. (*Answer* No. Symmetry dictates that E_x is zero at $x_P = 0$. However, Equation 22-5 gives a negative value for E_x at $x_P = 0$. These contradictory results cannot both be valid.)

$\vec{E}$ off the Axis of a Finite Line Charge

A charge Q is uniformly distributed on a straight-line segment of length L, as shown in Figure 22-3. We wish to find the electric field at an arbitrarily positioned field point P. To calculate the electric field at P we first choose coordinate axes. We choose the x axis through the line charge and the y axis through point P as shown. The ends of the charged line segment are labeled x_1 and x_2. A typical charge element $dq = \lambda\, dx$ that produces a field $d\vec{E}$ is shown in the figure. The field at P has both an x and a y component. Only the y component is computed here. (The x component is to be computed in Problem 22-27.)

The magnitude of the field produced by an element of charge $dq = \lambda\, dx$ is

$$|d\vec{E}| = \frac{k\,dq}{r^2} = \frac{k\lambda\,dx}{r^2}$$

and the y component is

$$dE_y = |d\vec{E}|\cos\theta = \frac{k\lambda\,dx}{r^2}\frac{y}{r} = \frac{k\lambda y\,dx}{r^3} \qquad \text{22-6}$$

where $\cos\theta = y/r$ and $r = \sqrt{x^2 + y^2}$. The total y component E_y is computed by integrating from $x = x_1$ to $x = x_2$.

$$E_y = \int_{x=x_1}^{x=x_2} dE_y = k\lambda y\int_{x_1}^{x_2}\frac{dx}{r^3} \qquad \text{22-7}$$

In calculating this integral y remains fixed. One way to execute this calculation is to use trigonometric substitution. From the figure we can see that $x = y\tan\theta$, so $dx = y\sec^2\theta\, d\theta$.† We also can see that $y = r\cos\theta$, so $1/r = \cos\theta/y$. Substituting these into Equation 22-7 gives

$$E_y = k\lambda y\frac{1}{y^2}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{k\lambda}{y}(\sin\theta_2 - \sin\theta_1) = \frac{kQ}{Ly}(\sin\theta_2 - \sin\theta_1) \qquad \text{22-8a}$$

E_y DUE TO A UNIFORMLY CHARGED LINE SEGMENT

EXERCISE Show that for the line charge shown in Figure 22-3 $dE_x = -k\lambda x dx/r^3$.

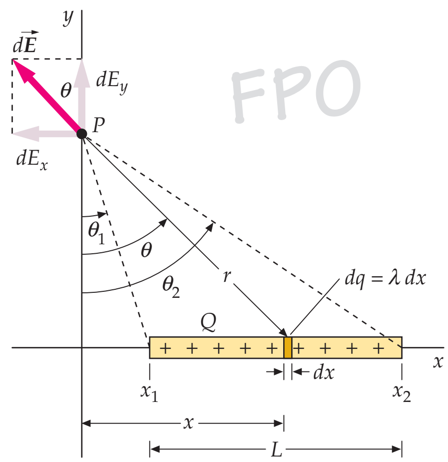


FIGURE 22-3 Geometry for the calculation of the electric field at field point P due to a uniform finite line charge.

† We have used the relation $d(\tan\theta)/d\theta = \sec^2\theta$.

The x component for the finite line charge shown in Figure 22-3 (and computed in Problem 22-27) is

$$E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1) \qquad \text{22-8}b$$

E_x DUE TO A UNIFORMLY CHARGED LINE SEGMENT

Electric field lines near a long wire. The electric field near a high-voltage power line can be large enough to ionize air, making the air a conductor. The glow resulting from the recombination of free electrons with the ions is called corona discharge.

$\vec{E}$ Due to an Infinite Line Charge

A line charge may be considered infinite if for any field point of interest P (see Figure 22-3), $x_1 \to -\infty$ and $x_2 \to +\infty$. We compute E_x and E_y for an infinite line charge using Equations 22-8*a* and *b* in the limit that $\theta_1 \to -\pi/2$ and $\theta_2 \to \pi/2$. (From Figure 22-3 we can see that this is the same as the limit that $x_1 \to -\infty$ and $x_2 \to +\infty$.) Substituting $\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$ into Equations 22-8*a* and *b* gives $E_x = 0$ and $E_y = \dfrac{2k\lambda}{y}$, where y is the perpendicular distance from the line charge to the field point. Thus,

$$E_R = 2k\frac{\lambda}{R} \qquad \text{22-9}$$

$\vec{E}$ AT A DISTANCE R FROM AN INFINITE LINE CHARGE

where R is the perpendicular distance from the line charge to the field point.

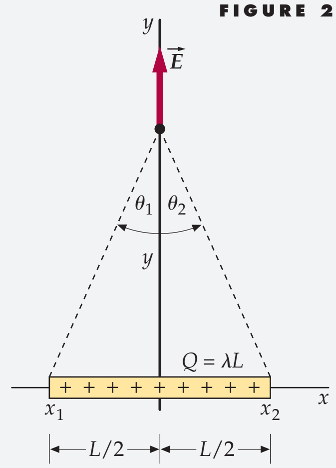
EXERCISE Show that Equation 22-9 has the correct units for the electric field.

ELECTRIC FIELD ON THE AXIS OF A FINITE LINE CHARGE **EXAMPLE 22-1**

Using Equations 22-8*a* and *b*, obtain an expression for the electric field on the perpendicular bisector of a uniformly charged line segment with linear charge density λ and length L.

PICTURE THE PROBLEM Sketch the line charge on the x axis with the y axis as its perpendicular bisector. According to Figure 22-4 this means choosing $x_1 = -\frac{1}{2}L$ and $x_2 = \frac{1}{2}L$ so $\theta_1 = -\theta_2$. Then use Equations 22-8*a* and 22-8*b* to find the electric field.

FIGURE 22-4



1. Sketch the charge configuration with the line charge on the x axis with the y axis as its perpendicular bisector. Show the field point on the positive y axis a distance y from the origin:

2. Use Equation 22-8*a* to find an expression for E_y. Simplify using $\theta_2 = -\theta_1 = \theta$:

$$E_y = \frac{k\lambda}{y}(\sin\theta_2 - \sin\theta_1) = \frac{k\lambda}{y}[\sin\theta - \sin(-\theta)]$$

$$= \frac{2k\lambda}{y}\sin\theta$$

3. Express $\sin\theta$ in terms of y and L and substitute into the step 2 result:

$$\sin\theta = \frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}$$

so

$$E_y = \frac{2k\lambda}{y}\,\frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}$$

4. Use Equation 22-8*b* to determine E_x:

$$E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1) = \frac{k\lambda}{y}[\cos\theta - \cos(-\theta)]$$
$$= \frac{k\lambda}{y}(\cos\theta - \cos\theta) = 0$$

5. Express the vector $\vec{E}$:

$$\vec{E} = E_x\hat{i} + E_y\hat{j} = \boxed{\frac{2k\lambda}{y}\frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}\hat{j}}$$

Electric Field Near and Far From a Finite Line Charge — EXAMPLE 22-2

A line charge of linear density $\lambda = 4.5$ nC/m lies on the x axis and extends from $x = -5$ cm to $x = 5$ cm. Using the expression for E_y obtained in Example 22-1, calculate the electric field on the y axis at (*a*) $y = 1$ cm, (*b*) $y = 4$ cm, and (*c*) $y = 40$ cm. (*d*) Estimate the electric field on the y axis at $y = 1$ cm, assuming the line charge to be infinite. (*e*) Find the total charge and estimate the field at $y = 40$ cm, assuming the line charge to be a point charge.

PICTURE THE PROBLEM Use the result of Example 22-1 to obtain the electric field on the y axis. In the expression for $\sin\theta_0$, we can express L and y in centimeters because the units cancel. (*d*) To find the field very near the line charge, we use $E_y = 2k\lambda/y$. (*e*) To find the field very far from the charge, we use $E_y = kQ/y^2$ with $Q = \lambda L$.

1. Calculate E_y at $y = 1$ cm for $\lambda = 4.5$ nC/m and $L = 10$ cm. We can express L and y in centimeters in the fraction on the right because the units cancel.

$$E_y = \frac{2k\lambda}{y}\frac{\frac{1}{2}L}{\sqrt{(\frac{1}{2}L)^2 + y^2}}$$
$$= \frac{2(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(4.5\times10^{-9}\ \text{C/m})}{0.01\ \text{m}}\frac{5\ \text{cm}}{\sqrt{(5\ \text{cm})^2 + (1\ \text{cm})^2}}$$
$$= \frac{80.9\ \text{N}\cdot\text{m/C}}{0.04\ \text{m}}\frac{5\ \text{cm}}{\sqrt{(5\ \text{cm})^2 + (1\ \text{cm})^2}} = 7.94\times10^3\ \text{N/C}$$
$$= \boxed{7.93\ \text{kN/C}}$$

2. Repeat the calculation for $y = 4$ cm $= 0.04$ m using the result $2k\lambda = 80.9$ N·m/C to simplify the notation:

$$E_y = \frac{80.9\ \text{N}\cdot\text{m/C}}{0.04\ \text{m}}\frac{5\ \text{cm}}{\sqrt{(5\ \text{cm})^2 + (4\ \text{cm})^2}} = 1.58\times10^3\ \text{N/C}$$
$$= \boxed{1.58\ \text{kN/C}}$$

3. Repeat the calculation for $y = 40$ cm:

$$E_y = \frac{80.9\ \text{N}\cdot\text{m/C}}{0.40\ \text{m}}\frac{5\ \text{cm}}{\sqrt{(5\ \text{cm})^2 + (40\ \text{cm})^2}} = \boxed{25.1\ \text{N/C}}$$

4. Calculate the field at $y = 1$ cm $= 0.01$ m due to an infinite line charge:

$$E_y \approx \frac{2k\lambda}{y} = \frac{80.9\ \text{N}\cdot\text{m/C}}{0.01\ \text{m}} = \boxed{8.09\ \text{kN/m}}$$

5. Calculate the total charge λL for $L = 0.1$ m and use it to find the field of a point charge at $y = 4$ m:

$$Q = \lambda L = (4.5\ \text{nC/m})(0.1\ \text{m}) = 0.45\ \text{nC}$$
$$E_y \approx \frac{k\lambda L}{y^2} = \frac{kQ}{y^2} = \frac{(8.99\times10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(0.45\times10^{-9}\ \text{C})}{(0.40\ \text{m})^2}$$
$$= \boxed{25.3\ \text{N/C}}$$

REMARKS At 1 cm from the 10-cm-long line charge, the estimated value of 8.09 kN/C obtained by assuming an infinite line charge differs from the exact value of 7.93 calculated in (*a*) by about 2 percent. At 40 cm from the line charge, the approximate value of 25.3 N/C obtained by assuming the line charge to be a point charge differs from the exact value of 25.1 N/C obtained in (*c*) by about 1 percent. Figure 22-5 shows the exact result for this line segment of length 10 cm and charge density 4.5 nC/m, and for the limiting cases of an infinite line charge of the same charge density, and a point charge $Q = \lambda L$.

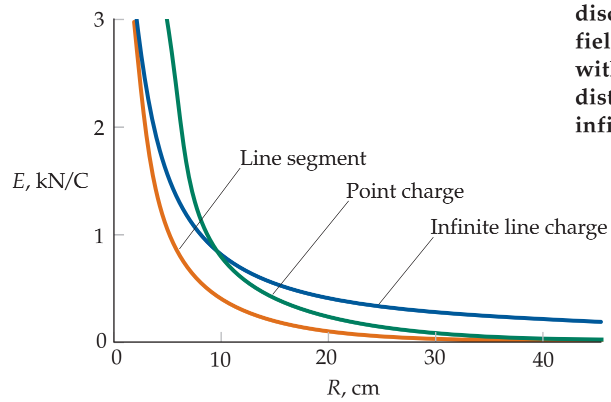


FIGURE 22-5 The magnitude of the electric field is plotted versus distance for the 10-cm-long line charge, the point charge, and the infinite line charge discussed in Example 22-2. Note that the field of the finite line segment converges with the field of the point charge at large distances, and with the field of the infinite line charge at small distances.

FIELD DUE TO A LINE CHARGE AND A POINT CHARGE — EXAMPLE 22-3 Try It Yourself

An infinitely long line charge of linear charge density λ = 0.6 μC/m lies along the z axis, and a point charge q = 8 μC lies on the y axis at y = 3 m. Find the electric field at the point P on the x axis at x = 4 m.

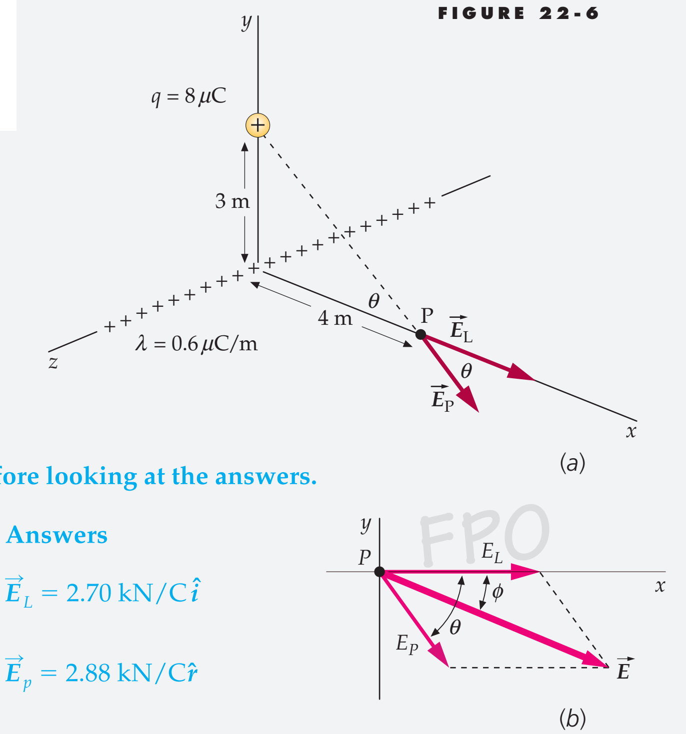


FIGURE 22-6

PICTURE THE PROBLEM The electric field for this system is the superposition of the fields due to the infinite line charge and the point charge. The field of the line charge, $\vec{E}_L$, points radially away from the z axis (Figure 22-6). Thus, at point P on the x axis, $\vec{E}_L$ is in the positive x direction. The point charge produces a field $\vec{E}_p$ along the line connecting q and the point P. The distance from q to P is $r = \sqrt{(3\text{ m})^2 + (4\text{ m})^2} = 5\text{ m}$.

Cover the column to the right and try these on your own before looking at the answers.

Steps	Answers
1. Calculate the field $\vec{E}_L$ at point P due to the infinite line charge.	$\vec{E}_L = 2.70\text{ kN/C}\,\hat{i}$
2. Find the field $\vec{E}_p$ at point P due to the point charge. Express $\vec{E}_p$ in terms of the unit vector $\hat{r}$ that points from q toward P.	$\vec{E}_p = 2.88\text{ kN/C}\,\hat{r}$
3. Find the x and y components of $\vec{E}_p$.	$E_{px} = E_p(0.8) = 2.30\text{ kN/C}$ $E_{py} = E_p(-0.6) = -1.73\text{ kN/C}$
4. Find the x and y components of the total field at point P.	$E_x = \boxed{5.00\text{ kN/C}}$, $E_y = \boxed{-1.73\text{ kN/C}}$
5. Use your result in step 4 to calculate the magnitude of the total field.	$E = \sqrt{E_x^2 + E_y^2} = \boxed{5.29\text{ kN/C}}$
6. Use your results in step 4 to find the angle ϕ between the field and the direction of increasing x.	$\phi = \tan^{-1}\dfrac{E_y}{E_x} = \boxed{-19.1°}$

$\vec{E}$ on the Axis of a Ring Charge

Figure 22-7*a* shows a uniform ring charge of radius a and total charge Q. The field $d\vec{E}$ at point P on the axis due to the charge element dq is shown in the figure. This field has a component dE_x directed along the axis of the ring and a component $dE_\perp$ directed perpendicular to the axis. The perpendicular components cancel in pairs, as can be seen in Figure 22-7*b*. From the symmetry of the charge distribution, we can see that the net field due to the entire ring must lie along the axis of the ring; that is, the perpendicular components sum to zero.

The axial component of the field due to the charge element shown is

$$dE_x = \frac{k\,dq}{r^2}\cos\theta = \frac{k\,dq}{r^2}\frac{x}{r} = \frac{k\,dq\,x}{(x^2+a^2)^{3/2}}$$

where

$$r^2 = x^2 + a^2 \quad \text{and} \quad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+a^2}}$$

The field due to the entire ring of charge is

$$E_x = \int \frac{kx\,dq}{(x^2+a^2)^{3/2}}$$

Since x does not vary as we integrate over the elements of charge, we can factor any function of x from the integral. Then

$$E_x = \frac{kx}{(x^2+a^2)^{3/2}}\int dq$$

or

$$E_x = \frac{kQx}{(x^2+a^2)^{3/2}} \qquad 22\text{-}10$$

A plot of E_x versus x along the axis of the ring is shown in Figure 22-8.

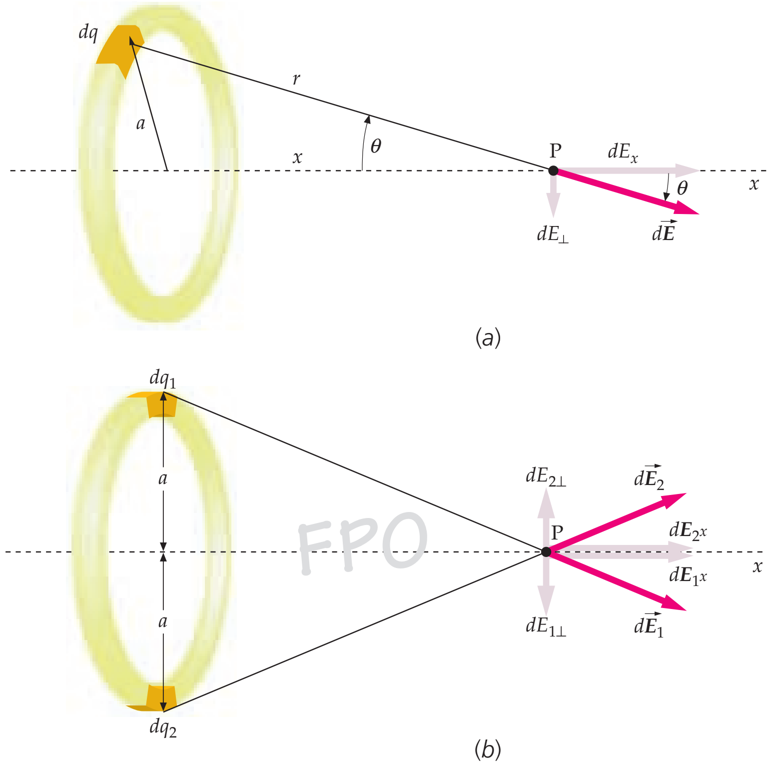


FIGURE 22-7 (*a*) A ring charge of radius a. The electric field at point P on the x axis due to the charge element dq shown has one component along the x axis and one perpendicular to the x axis. (*b*) For any charge element dq_1 there is an equal charge element dq_2 opposite it, and the electric-field components perpendicular to the x axis sum to zero.

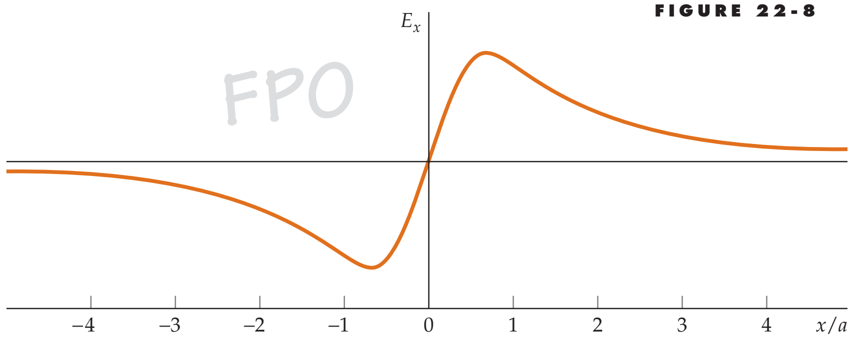


FIGURE 22-8

EXERCISE Find the point on the axis of the ring where E_x is maximum. (*Answer* $x = a/\sqrt{2}$)

$\vec{E}$ on the Axis of a Uniformly Charged Disk

Figure 22-9 shows a uniformly charged disk of radius R and total charge Q. We can calculate the field on the axis of the disk by treating the disk as a set of concentric ring charges. Let the axis of the disk be the x axis. $\vec{E}$ due to the charge on each ring is along the x axis. A ring of radius a and width da is shown in the figure. The area of this ring is $dA = 2\pi a\,da$, and its charge is $dq = \sigma\,dA = 2\pi\sigma a\,da$, where $\sigma = Q/\pi R^2$ is the surface charge density (the charge per unit area). The field produced by this ring is given by Equation 22-10 if we replace Q with $dq = 2\pi\sigma a\,da$.

$$dE_x = \frac{kx2\pi\sigma a\,da}{(x^2+a^2)^{3/2}}$$

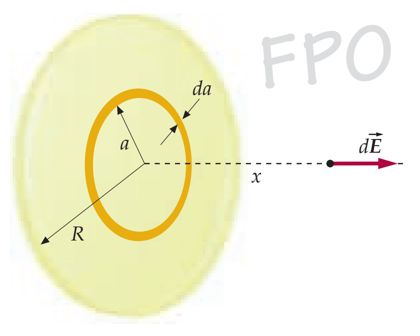


FIGURE 22-9 A uniform disk of charge can be treated as a set of ring charges, each of radius a.

The total field is found by integrating from $a = 0$ to $a = R$:

$$E_x = \int_0^R \frac{kx2\pi\sigma a\, da}{(x^2 + a^2)^{3/2}} = kx\pi\sigma \int_0^R (x^2 + a^2)^{-3/2} 2a\, da = kx\pi\sigma \int_{x^2+0^2}^{x^2+R^2} u^{-3/2} du$$

where $u = x^2 + a^2$, so $du = 2a\, da$. The integration thus gives

$$E_x = kx\pi\sigma \frac{u^{-1/2}}{-1/2}\bigg|_{x^2}^{x^2+R^2} = -2kx\pi\sigma\left(\frac{1}{\sqrt{x^2 + R^2}} - \frac{1}{\sqrt{x^2}}\right)$$

This can be expressed

$$E_x = 2\pi k\sigma\left(1 - \frac{1}{\sqrt{1 + \dfrac{R^2}{x^2}}}\right), \qquad x > 0 \qquad \text{22-11}$$

$\vec{E}$ ON THE AXIS OF A DISK CHARGE

EXERCISE Find an expression for E_x on the negative x axis. (*Answer* $E_x = -2\pi k\sigma\left(1 - \dfrac{1}{\sqrt{1 + \dfrac{R^2}{x^2}}}\right)$ for $x < 0$)

For $x >> R$ (on the positive x axis far from the disk) we expect it to look like a point charge. If we merely replace R^2/x^2 with 0 for $x >> R$, we get $E_x \to 0$. Although this is correct, it does not tell us anything about how E_x depends on x for large x. We can find this dependence by using the binomial expansion, $(1 + \epsilon)^n \approx 1 + n\epsilon$, for $|\epsilon| << 1$. Using this approximation on the second term in Equation 22-11, we obtain

$$\frac{1}{\left(1 + \dfrac{R^2}{x^2}\right)^{1/2}} = \left(1 + \frac{R^2}{x^2}\right)^{-1/2} \approx 1 - \frac{R^2}{2x^2}$$

Substituting this into Equation 22-11 we obtain

$$E_x \approx 2\pi k\sigma\left(1 - 1 + \frac{R^2}{2x^2}\right) = \frac{k\pi R^2\sigma}{2x^2} = \frac{kQ}{x^2}, \qquad x >> R \qquad \text{22-12}$$

where $Q = \sigma\pi R^2$ is the total charge on the disk. For large x, the electric field of the charged disk approaches that of a point charge Q at the origin.

$\vec{E}$ Due to an Infinite Plane of Charge

The field of an infinite plane of charge can be obtained from Equation 22-11 by letting the ratio R/x go to infinity. Then

$$E_x = 2\pi k\sigma, \qquad x > 0 \qquad \text{22-13a}$$

$\vec{E}$ NEAR AN INFINITE PLANE OF CHARGE

Thus, the field due to an infinite-plane charge distribution is uniform; that is, the field does not depend on x. On the other side of the infinite plane, for negative values of x, the field points in the negative x direction, so

$$E_x = -2\pi k\sigma, \qquad x < 0 \qquad \text{22-13b}$$

As we move along the x axis, the electric field jumps from $-2\pi k\sigma\hat{i}$ to $+2\pi k\sigma\hat{i}$ when we pass through an infinite plane of charge (Figure 22-10). There is thus a discontinuity in E_x in the amount $4\pi k\sigma$.

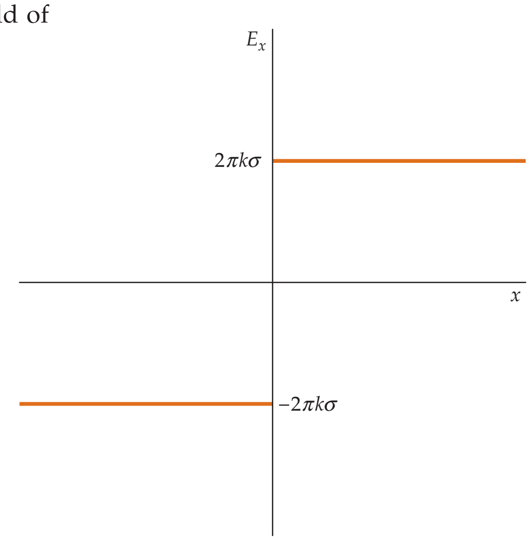


FIGURE 22-10 **Graph showing the discontinuity of $\vec{E}$ at a plane charge.**

Electric Field on the Axis of a Disk — EXAMPLE 22-4

A disk of radius 5 cm carries a uniform surface charge density of 4 μC/m^2. Using appropriate approximations, find the electric field on the axis of the disk at distances of (*a*) 0.01 cm, (*b*) 0.03 cm, and (*c*) 6 m. (*d*) Compare the results for (*a*), (*b*), and (*c*) with the exact values arrived at by using Equation 22-11.

PICTURE THE PROBLEM For the comparisons in Part (*d*), we will carry out all calculations to five-figure accuracy. For (*a*) and (*b*), the field point is very near the disk compared with its radius, so we can approximate the disk as an infinite plane. For (*c*), the field point is sufficiently far from the disk ($x/R = 120$) that we can approximate the disk as a point charge. (*d*) To compare, we find the percentage difference between the approximate values and the exact values.

(*a*) The electric field near the disk is approximately that due to an infinite plane charge:

$$E_x \approx 2\pi k\sigma$$
$$= 2\pi(8.98755 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)(4 \times 10^6\ \text{C/m}^2)$$
$$= \boxed{225.88\ \text{kN/C}}$$

(*b*) Since 0.03 cm is still very near the disk, the disk still looks like an infinite plane charge:

$$E_x \approx 2\pi k\sigma = \boxed{225.88\ \text{kN/C}}$$

(*c*) Far from the disk, the field is approximately that due to a point charge:

$$E_x \approx \frac{kQ}{x^2} = \frac{k\sigma\pi R^2}{x^2} = 2\pi k\sigma\frac{R^2}{2x^2}$$
$$= (225.88\ \text{kN/C})\frac{(0.05\ \text{m})^2}{2(6\ \text{m})^2} = \boxed{7.8431\ \text{N/C}}$$

(*d*) Using the exact expression (Equation 22-11) for E_x, we calculate the exact values at the specified points:

$$E_x(\text{exact}) = 2\pi k\sigma\left(1 - \frac{1}{\sqrt{1 + \frac{R^2}{x^2}}}\right)$$

x (cm)	E_x (exact) (N/C)	E_x (approx) (N/C)	% diff
0.01	225,430	225,880	0.2
0.03	224,530	225,880	0.6
600	7.8427	7.8431	0.005

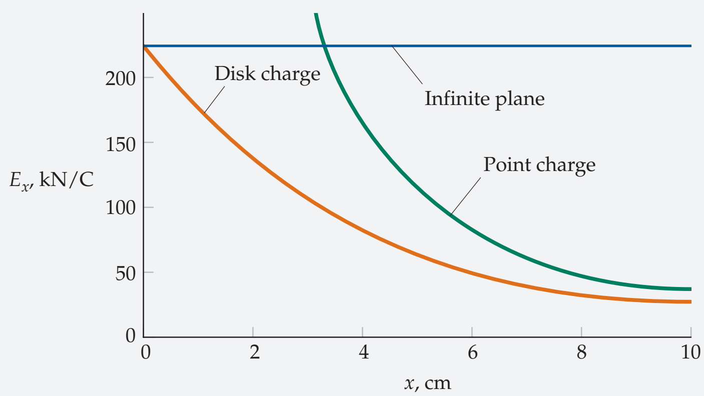


FIGURE 22-11 Note that the field of the disk charge converges with the field of the point charge at large distances, and equals the field of the infinite plane charge in the limit that x approaches zero.

REMARKS Figure 22-11 shows E_x versus x for the disk charge in this example, for an infinite plane with the same charge density, and for a point charge.

22-2 Gauss's Law

In Chapter 21, the electric field is described visually via electric field lines. Here that description is put in rigorous mathematical language called Gauss's law. Gauss's law is one of Maxwell's equations—the fundamental equations of electromagnetism, which are the topic of Chapter 31. For static charges, Gauss's law and Coulomb's law are equivalent. Electric fields arising from some symmetrical charge distributions, such as a spherical shell of charge or an infinite line of charge, can be easily calculated using Gauss's law. In this section, we give an argument for the validity of Gauss's law based on the properties of electric field lines. A rigorous derivation of Gauss's law is presented in Section 22-6.

A closed surface is one that divides the universe into two distinct regions, the region inside the surface and the region outside the surface. Figure 22-12 shows a closed surface of arbitrary shape enclosing a dipole. The number of electric field lines beginning on the positive charge and penetrating the surface from the inside depends on where the surface is drawn, but any line penetrating the surface from the inside also penetrates it from the outside. To count the net number of lines out of any closed surface, count any line that penetrates from the inside as $+1$, and any penetration from the outside as -1. Thus, for the surface shown (Figure 22-12), the net number of lines out of the surface is zero. For surfaces enclosing other types of charge distributions, such as that shown in Figure 22-13, *the net number of lines out of any surface enclosing the charges is proportional to the net charge enclosed by the surface.* This rule is a qualitative statement of Gauss's law.

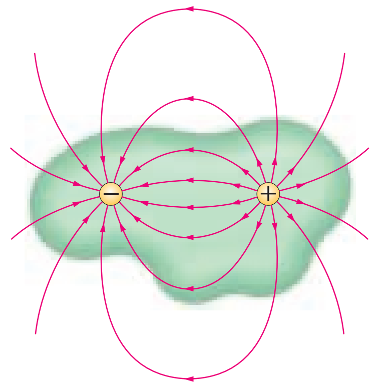


FIGURE 22-12 A surface of arbitrary shape enclosing an electric dipole. As long as the surface encloses both charges, the number of lines penetrating the surface from the inside is exactly equal to the number of lines penetrating the surface from the outside no matter where the surface is drawn.

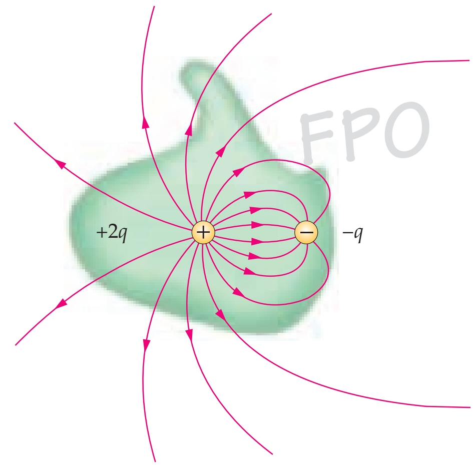


FIGURE 22-13 A surface of arbitrary shape enclosing the charges $+2q$ and $-q$. Either the field lines that end on $-q$ do not pass through the surface or they penetrate it from the inside the same number of times as from the outside. The net number that exit, the same as that for a single charge of $+q$, is equal to the net charge enclosed by the surface.

Electric Flux

The mathematical quantity that corresponds to the number of field lines penetrating a surface is called the **electric flux** ϕ. For a surface perpendicular to $\vec{E}$ (Figure 22-14), the electric flux is the product of the magnitude of the field E and the area A:

$$\phi = EA$$

The units of flux are N·m²/C. Because E is proportional to the number of field lines per unit area, the flux is proportional to the number of field lines penetrating the surface.

In Figure 22-15, the surface of area A_2 is not perpendicular to the electric field $\vec{E}$. However, the number of lines that penetrate the surface of area A_2 is the same as the number that penetrate the surface of area A_1, which is perpendicular to $\vec{E}$. These areas are related by

$$A_2 \cos \theta = A_1 \tag{22-14}$$

where θ is the angle between $\vec{E}$ and the unit vector $\hat{n}$ that is normal to the surface A_2, as shown in the figure. The electric flux through a surface is defined to be

$$\phi = \vec{E} \cdot \hat{n}A = EA \cos \theta = E_n A \tag{22-15}$$

where $E_n = \vec{E} \cdot \hat{n}$ is the component of $\vec{E}$ normal (perpendicular) to the surface.

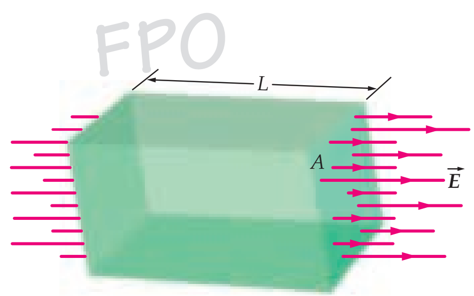


FIGURE 22-14 Electric field lines of a uniform field penetrating a surface of area A that is oriented perpendicular to the field. The product EA is the electric flux through the surface.

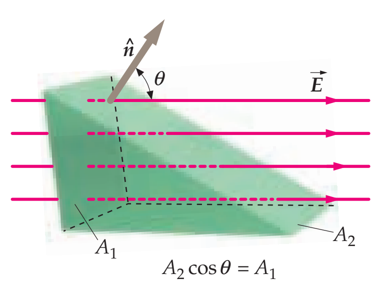


FIGURE 22-15 Electric field lines of a uniform electric field that is perpendicular to the surface of area A_1 but makes an angle θ with the unit vector $\hat{n}$ that is normal to the surface of area A_2. Where $\vec{E}$ is not perpendicular to the surface, the flux is $E_n A$, where $E_n = E \cos \theta$ is the component of $\vec{E}$ that is perpendicular to the surface. The flux through the surface of area A_2 is the same as that through the surface of area A_1.

Figure 22-16 shows a curved surface over which $\vec{E}$ may vary. If the area ΔA_i of the surface element that we choose is small enough, it can be considered to be a plane, and the variation of the electric field across the element can be neglected. The flux of the electric field through this element is

$$\Delta\phi_i = E_{ni}\,\Delta A_i = \vec{E}_i \cdot \hat{n}_i\,\Delta A_i$$

where $\hat{n}_i$ is the unit vector perpendicular to the surface element and $\vec{E}_i$ is the electric field anywhere on the surface element. If the surface is curved, the unit vectors for different elements will have different directions. The total flux through the surface is the sum of $\Delta\phi_i$ over all the elements making up the surface. In the limit, as the number of elements approaches infinity and the area of each element approaches zero, this sum becomes an integral. The general definition of electric flux is thus:

$$\phi = \lim_{\Delta A_i \to 0} \sum_i \vec{E}_i \cdot \hat{n}_i\,\Delta A_i = \int_S \vec{E} \cdot \hat{n}\,dA \qquad \text{22-16}$$

DEFINITION—ELECTRIC FLUX

where the S stands for the surface we are integrating over.

On a *closed* surface we are interested in the electric flux out of the surface, so we choose the unit vector $\hat{n}$ to be outward at each point. The integral over a closed surface is indicated by the symbol $\oint$. The total or net flux out of a closed surface is therefore written

$$\phi_{\text{net}} = \oint_S \vec{E} \cdot \hat{n}\,dA = \oint_S E_n\,dA \qquad \text{22-17}$$

The net flux ϕ_{net} through the closed surface is positive or negative, depending on whether $\vec{E}$ is predominantly outward or inward at the surface. At points on the surface where $\vec{E}$ is inward, E_n is negative.

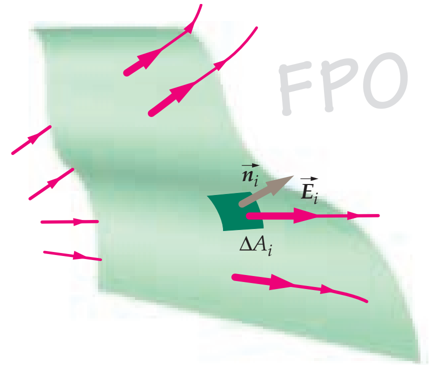


FIGURE 22-16 If E_n varies from place to place on a surface, either because E varies or because the angle between $\vec{E}$ and $\hat{n}$ varies, the area of the surface is divided into small elements of area ΔA_i. The flux through the surface is computed by summing $\vec{E}_i \cdot \hat{n}_i\,\Delta A_i$ over all the area elements.

Quantitative Statement of Gauss's Law

Figure 22-17 shows a spherical surface of radius R with a point charge Q at its center. The electric field everywhere on this surface is normal to the surface and has the magnitude

$$E_n = \frac{kQ}{R^2}$$

The net flux of $\vec{E}$ out of this spherical surface is

$$\phi_{\text{net}} = \oint_S E_n\,dA = E_n \oint_S dA$$

where we have taken E_n out of the integral because it is constant everywhere on the surface. The integral of dA over the surface is just the total area of the surface, which for a sphere of radius R is $4\pi R^2$. Using this and substituting kQ/R^2 for E_n, we obtain

$$\phi_{\text{net}} = \frac{kQ}{R^2}4\pi R^2 = 4\pi kQ \qquad \text{22-18}$$

Thus, the net flux out of a spherical surface with a point charge at its center is independent of the radius R of the sphere and is equal to $4\pi k$ times Q (the point charge). This is consistent with our previous observation that the net number of

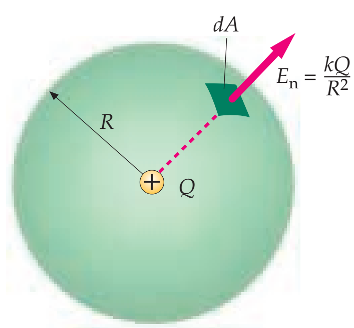


FIGURE 22-17 A spherical surface enclosing a point charge Q. (*a*) The net number of electric field lines out of this surface and the net number out of any surface that also encloses Q is the same. (*b*) The net flux is easily calculated for a spherical surface. It equals E_n times the surface area, or $E_n 4\pi R^2$.

lines going out of a closed surface is proportional to the net charge inside the surface. *This number of lines is the same for all closed surfaces surrounding the charge, independent of the shape of the surface.* Thus, the net flux out of *any surface* surrounding a point charge Q equals $4\pi kQ$.

We can extend this result to systems containing multiple charges. In Figure 22-18, the surface encloses two point charges, q_1 and q_2, and there is a third point charge q_3 outside the surface. Since the electric field at any point on the surface is the vector sum of the electric fields produced by each of the three charges, the net flux $\phi_{\text{net}} = \oint_s \vec{E} \cdot \hat{n}\, dA$ out of the surface is just the sum of the fluxes due to the individual charges. The flux due to charge q_3, which is outside the surface, is zero because every field line from q_3 that enters the surface at one point leaves the surface at some other point. The flux out of the surface due to charge q_1 is $4\pi kq_1$ and that due to charge q_2 is $4\pi kq_2$. The net flux out of the surface therefore equals $4\pi k(q_1 + q_2)$, which may be positive, negative, or zero depending on the signs and magnitudes of q_1 and q_2.

FIGURE 22-18 **A surface enclosing point charges q_1 and q_2, but not q_3. The net flux out of this surface is $4\pi k(q_1 + q_2)$.**

The net outward flux through any closed surface equals $4\pi k$ times the net charge inside the surface:

$$\phi_{net} = \oint_S E_n\, dA = 4\pi k Q_{\text{inside}} \tag{22-19}$$

GAUSS'S LAW

This is **Gauss's law.** Its validity depends on the fact that the electric field due to a single point charge varies inversely with the square of the distance from the charge. It was this property of the electric field that made it possible to draw a fixed number of electric field lines from a charge and have the density of lines be proportional to the field strength.

It is customary to write the Coulomb constant k in terms of another constant ϵ_0, which is called the **permittivity of free space:**

$$k = \frac{1}{4\pi\, \epsilon_0} \tag{22-20}$$

Using this notation, Coulomb's law for $\vec{E}$ is written

$$\vec{E} = \frac{1}{4\pi\, \epsilon_0} \frac{q}{r^2}\hat{r} \tag{22-21}$$

and Gauss's law is written

$$\phi_{\text{net}} = \oint_S E_n\, dA = \frac{Q_{\text{inside}}}{\epsilon_0} \tag{22-22}$$

The value of ϵ_0 in SI units is

$$\epsilon_0 = \frac{1}{4\pi k} = \frac{1}{4\pi(8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2)} = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2 \tag{22-23}$$

Gauss's law is valid for all surfaces and all charge distributions. For charge distributions that have high degrees of symmetry, it can be used to calculate the electric field, as we illustrate in the next section. For static charge distributions, Gauss's law and Coulomb's law are equivalent. However, Gauss's law is more general in that it is always valid and Coulomb's law is valid only for static charge distributions.

FLUX THROUGH AND CHARGE INSIDE AN IMAGINARY CLOSED SURFACE

EXAMPLE 22-5

An electric field is $\vec{E} = (200\text{ N/C})\hat{i}$ in the region $x > 0$ and $\vec{E} = (-200\text{ N/C})\hat{i}$ in the region $x < 0$. An imaginary soup-can shaped surface of length 20 cm and radius $R = 5$ cm has its center at the origin and its axis along the x axis, so that one end is at $x = +10$ cm and the other is at $x = -10$ cm (Figure 22-19). (*a*) What is the net outward flux through the entire closed surface? (*b*) What is the net charge inside the closed surface?

PICTURE THE PROBLEM The closed surface described, which is piecewise continuous, consists of three pieces—two flat ends and a curved side. Separately calculate the flux of $\vec{E}$ out of each piece of the surface. To calculate the flux out of a piece draw the outward normal $\hat{n}$ at a randomly chosen point on the piece and draw the vector $\vec{E}$ at the same point. If $E_n = \vec{E} \cdot \hat{n}$ is the same everywhere on the piece, then the outward flux through it is $\phi = \vec{E} \cdot \hat{n}A$ (Equation 22-15). The net outward flux through the entire closed surface is obtained by summing the fluxes through the individual pieces. The net outward flux is related to the charge inside by Gauss's law (Equation 22-19).

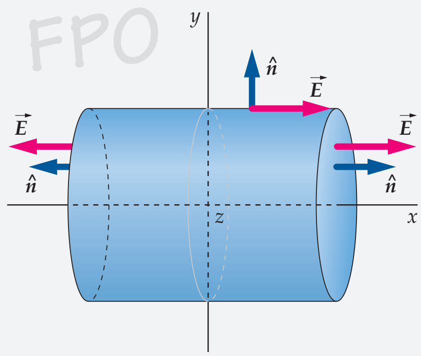


FIGURE 22-19

(*a*) 1. Sketch the soup-can shaped surface. On each piece of the surface draw the outward normal $\hat{n}$ and the vector $\vec{E}$:

2. Calculate the outward flux through the right circular flat surface where $\hat{n} = \hat{i}$:

$$\phi_{\text{right}} = \vec{E}_{\text{right}} \cdot \hat{n}_{\text{right}}A = \vec{E}_{\text{right}} \cdot \hat{i}\pi R^2$$
$$= (200\text{ N/C})\hat{i} \cdot \hat{i}(\pi)(0.05\text{ m})^2$$
$$= 1.57\text{ N}\cdot\text{m}^2/\text{C}$$

3. Calculate the outward flux through the left circular surface where $\hat{n} = -\hat{i}$:

$$\phi_{\text{left}} = \vec{E}_{\text{left}} \cdot \hat{n}_{\text{left}}A = \vec{E}_{\text{left}} \cdot (-\hat{i})\pi R^2$$
$$= (-200\text{ N/C})\hat{i} \cdot (-\hat{i})(\pi)(0.05\text{ m})^2$$
$$= 1.57\text{ N}\cdot\text{m}^2/\text{C}$$

4. Calculate the outward flux through the curved surface where $\vec{E}$ is perpendicular to $\hat{n}$:

$$\phi_{\text{curved}} = \vec{E}_{\text{curved}} \cdot \hat{n}_{\text{curved}}A = 0$$

5. The net outward flux is the sum through all the individual surfaces:

$$\phi_{\text{net}} = \phi_{\text{right}} + \phi_{\text{left}} + \phi_{\text{curved}}$$
$$= 1.57\text{ N}\cdot\text{m}^2/\text{C} + 1.57\text{ N}\cdot\text{m}^2/\text{C} + 0$$
$$= \boxed{3.14\text{ N}\cdot\text{m}^2/\text{C}}$$

(*b*) Gauss's law relates the charge inside to the net flux:

$$Q_{\text{inside}} = \epsilon_0\,\phi_{\text{net}}$$
$$= (8.85 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)(3.14 \times \text{N}\cdot\text{m}^2/\text{C})$$
$$= \boxed{2.78 \times 10^{-11}\text{ C} = 27.8\text{ pC}}$$

REMARKS The flux does not depend on the length of the can. This means the charge inside resides entirely on the yz plane.

22-3 Calculating $\vec{E}$ From Gauss's Law

Given a highly symmetrical charge distribution, the electric field can often be calculated more easily using Gauss's law than it can be using Coulomb's law. We first find an imaginary closed surface, called a **Gaussian surface** (the soup can in Example 22-5). Optimally, this surface is chosen so that on each of its pieces $\vec{E}$ is either zero, perpendicular to $\hat{n}$, or parallel to $\hat{n}$ with E_n constant. Then the flux through each piece equals E_nA and Gauss's law is used to relate the field to the charges inside the closed surface.

Plane Symmetry

A charge distribution has **plane symmetry** if the views of it from all points on an infinite plain surface are the same. Figure 22-20 shows an infinite plane of charge of uniform surface charge density σ. By symmetry, $\vec{E}$ must be perpendicular to the plane and can depend only on the distance from it. Also, $\vec{E}$ must have the same magnitude but the opposite direction at points the same distance from the charged plane on either side of the plane. For our Gaussian surface, we choose a soup-can shaped cylinder as shown, with the charged plane bisecting the cylinder. On each piece of the cylinder is drawn both $\hat{n}$ and $\vec{E}$. Since $\vec{E}\cdot\hat{n}$ is zero everywhere on the curved piece of the Gaussian surface, there is no flux through it. The flux through each flat piece of the surface is E_nA, where A is the area of each flat piece. Thus, the total outward flux through the closed surface is $2E_nA$. The net charge inside the surface is σA. Gauss's law then gives

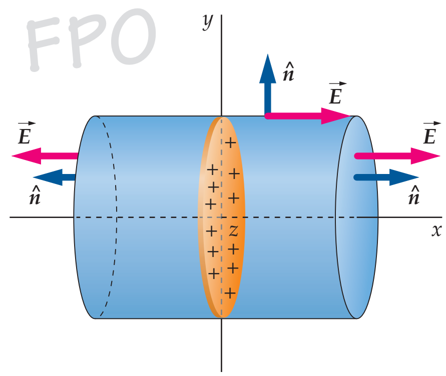


FIGURE 22-20 Gaussian surface for the calculation of $\vec{E}$ due to an infinite plane of charge. (Only the part of the plane that is inside the Gaussian surface is shown.) On the flat faces of this soup can, $\vec{E}$ is perpendicular to the surface and constant in magnitude. On the curved surface $\vec{E}$ is parallel with the surface.

$$Q_{\text{inside}} = \epsilon_0\,\phi_{\text{net}}$$

$$\sigma A = \epsilon_0\,2E_nA$$

(Can you see why $Q_{\text{inside}} = \sigma A$?) Solving for E_n gives

$$E_n = \frac{\sigma}{2\epsilon_0} = 2\pi k\sigma \qquad 22\text{-}24$$

$\vec{E}$ FOR AN INFINITE PLANE OF CHARGE

E_n is positive if σ is positive, and E_n is negative if σ is negative. This means if σ is positive $\vec{E}$ is directed away from the charged plane, and if σ is negative $\vec{E}$ points toward it. This is the same result that we obtained, with much more difficulty, using Coulomb's law (Equations 22-13*a* and *b*). Note that the field is discontinuous at the charged plane. If the charged plane is the yz plane, the field is $\vec{E} = \sigma/(2\epsilon_0)\hat{i}$ in the region $x > 0$ and $\vec{E} = -\sigma/(2\epsilon_0)\hat{i}$ in the region $x < 0$. Thus, the field is discontinuous by $\Delta\vec{E} = \sigma/(2\epsilon_0)\hat{i} - [-\sigma/(2\epsilon_0)\hat{i}] = (\sigma/\epsilon_0)\hat{i}$.

Electric Field Due to Two Infinite Planes **EXAMPLE 22-6**

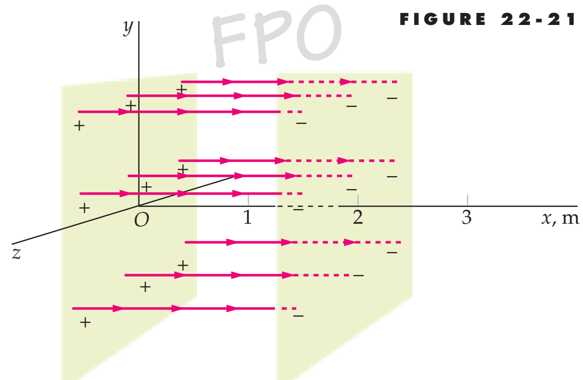


FIGURE 22-21

In Figure 22-21, an infinite plane of surface charge density $\sigma = +4.5\ \text{nC/m}^2$ lies in the $x = 0$ plane, and a second infinite plane of surface charge density $\sigma = -4.5\ \text{nC/m}^2$ lies in a plane parallel to the $x = 0$ plane at $x = 2$ m. Find the electric field at (*a*) $x = 1.8$ m and (*b*) $x = 5$ m.

PICTURE THE PROBLEM Each plane produces a uniform electric field of magnitude $E = \sigma/(2\epsilon_0)$. We use superposition to find the resultant field. Between the planes the fields add, producing a net field of magnitude $\sigma/(2\epsilon_0)$ in the positive x direction. For $x > 2$ m and for $x < 0$, the fields point in opposite directions and cancel.

(*a*) 1. Calculate the magnitude of the field E produced by each plane:

$$E = \frac{\sigma}{2\,\epsilon_0} = \frac{4.5\times10^{-9}\ \text{C/M}^2}{2(8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)}$$

$$= 254\ \text{N/C}$$

2. At $x = 1.8$ m, between the planes, the field due to each plane points in the positive x direction:

$$E_{x,\text{net}} = E_1 + E_2 = 254\ \text{N/C} + 254\ \text{N/C}$$
$$= \boxed{508\ \text{N/C}}$$

(*b*) At $x = 5$ m, the fields due to the two planes are oppositely directed:

$$E_{x,\text{net}} = E_1 - E_2 = \boxed{0}$$

REMARKS Because the two planes carry equal and opposite charge densities, the electric field lines originate on the positive plane and terminate on the negative plane. $\vec{E}$ is zero except between the planes. Note that $E_{x,\text{net}} = 508$ N/C not just at $x = 1.8$ m but at any point in the region between the charged planes.

Spherical Symmetry

Assume a charge distribution is concentric within a spherical surface. The charge distribution has **spherical symmetry** if the views of it from all points on the spherical surface are the same. To calculate the electric field due to spherically symmetric charge distributions, we use a spherical surface for our Gaussian surface. We illustrate this by first finding the electric field at a distance r from a point charge q. We choose a spherical surface of radius r, centered at the point charge, for our Gaussian surface. By symmetry, $\vec{E}$ must be directed either radially outward or radially inward. It follows that the component of $\vec{E}$ normal to the surface equals the radial component of E at each point on the surface. That is, $E_n = \vec{E} \cdot \hat{n} = E_r$, where $\hat{n}$ is the outward normal, has the same value everywhere on the spherical surface. Also, the magnitude of $\vec{E}$ can depend on the distance from the charge but not on the direction from the charge. The net flux through the spherical surface of radius r is thus

$$\phi_{\text{net}} = \oint_S \vec{E} \cdot \hat{n}\, dA = \oint_S E_r\, dA = E_r \oint_S dA = E_r 4\pi r^2$$

where $\oint_s dA = 4\pi r^2$ the total area of the spherical surface. Since the total charge inside the surface is just the point charge q, Gauss's law gives

$$E_r 4\pi r^2 = \frac{q}{\epsilon_0}$$

Solving for E_r gives

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

which is Coulomb's law. We have thus derived Coulomb's law from Gauss's law. Because Gauss's law can also be derived from Coulomb's law (see Section 22-6), we have shown that the two laws are equivalent for static charges.

$\vec{E}$ Due to a Thin Spherical Shell of Charge

Consider a uniformly charged thin spherical shell of radius R and total charge Q. By symmetry, $\vec{E}$ must be radial, and its magnitude can depend only on the distance r from the center of the sphere. In Figure 22-22, we have chosen a spherical Gaussian surface of radius $r > R$. Since $\vec{E}$ is normal to this surface, and has the same magnitude everywhere on the surface, the flux through the surface is

$$\phi_{\text{net}} = \oint_S E_r\, dA = E_r \oint_S dA = E_r 4\pi r^2$$

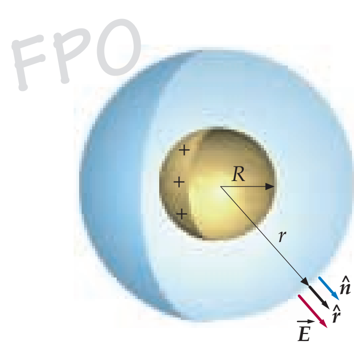


FIGURE 22-22 Spherical Gaussian surface of radius $r > R$ for the calculation of the electric field outside a uniformly charged thin spherical shell of radius R.

Since the total charge inside the Gaussian surface is the total charge on the shell Q, Gauss's law gives

$$E_r 4\pi r^2 = \frac{Q}{\epsilon_0}$$

or

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad r > R \qquad \text{22-25}a$$

$\vec{E}$ OUTSIDE A SPHERICAL SHELL OF CHARGE

Thus, the electric field outside a uniformly charged spherical shell is the same as if all the charge were at the center of the shell.

If we choose a spherical Gaussian surface inside the shell, where $r < R$, the net flux is again $E_r\,4\pi r^2$, but the total charge inside the surface is zero. Therefore, for $r < R$, Gauss's law gives

$$\phi_{\text{net}} = E_r 4\pi r^2 = 0$$

so

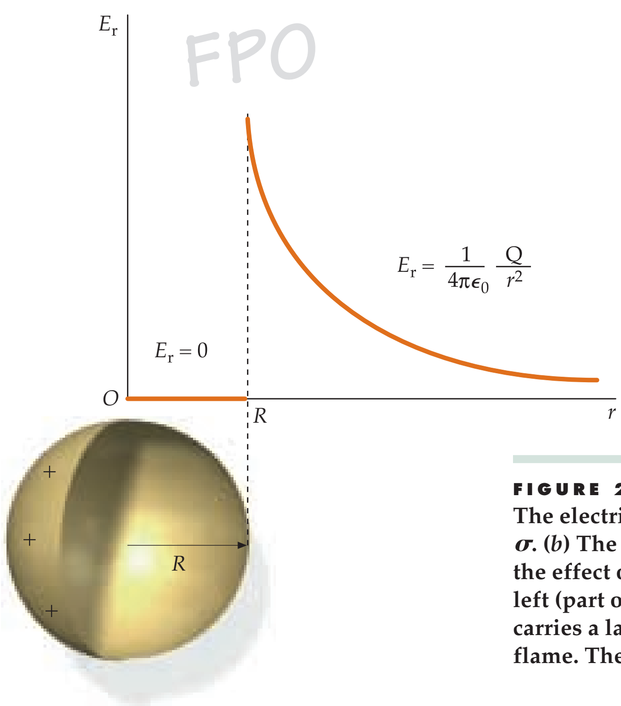
$$E_r = 0, \qquad r < R \qquad \text{22-25}b$$

$\vec{E}$ INSIDE A SPHERICAL SHELL OF CHARGE

These results can also be obtained by direct integration of Coulomb's law, but that calculation is much more difficult.

Figure 22-23 shows E_r versus r for a spherical-shell charge distribution. Again, note that the electric field is discontinuous at $r = R$, where the surface charge density is $\sigma = Q/4\pi R^2$. Just outside the shell at $r \approx R$, the electric field is $E_r = Q/4\pi\epsilon_0 R^2 = \sigma/\epsilon_0$, since $\sigma = Q/4\pi R^2$. Because the field just inside the shell is zero, the electric field is discontinuous by the amount σ/ϵ_0 as we pass through the shell.

(*a*)



(*b*)

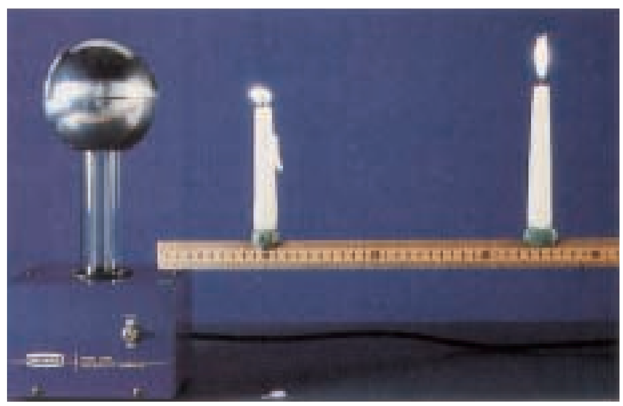


FIGURE 22-23 **(*a*) A plot of E_r versus r for a spherical-shell charge distribution. The electric field is discontinuous at $r = R$, where there is a surface charge of density σ. (*b*) The decrease in E_r over distance due to a charged spherical shell is evident by the effect of the field on the flames of these two candles. The spherical shell at the left (part of a Van de Graaff generator, a device that is discussed in Chapter 24) carries a large negative charge that attracts the positive ions in the nearby candle flame. The flame at right, which is much farther away, is not noticeably affected.**

ELECTRIC FIELD DUE TO A POINT CHARGE AND A CHARGED SPHERICAL SHELL

EXAMPLE 22-7

A spherical shell of radius $R = 3$ m has its center at the origin and carries a surface charge density of $\sigma = 3$ nC/m². A point charge $q = 250$ nC is on the y axis at $y = 2$ m. Find the electric field on the x axis at (a) $x = 2$ m and (b) $x = 4$ m.

PICTURE THE PROBLEM We find the field due to the point charge and that due to the spherical shell and sum the field vectors. For (a), the field point is inside the shell, so the field is due only to the point charge (Figure 22-24a). For (b), the field point is outside the shell, so the shell can be considered as a point charge at the origin. We then find the field due to two point charges (Figure 22-24b).

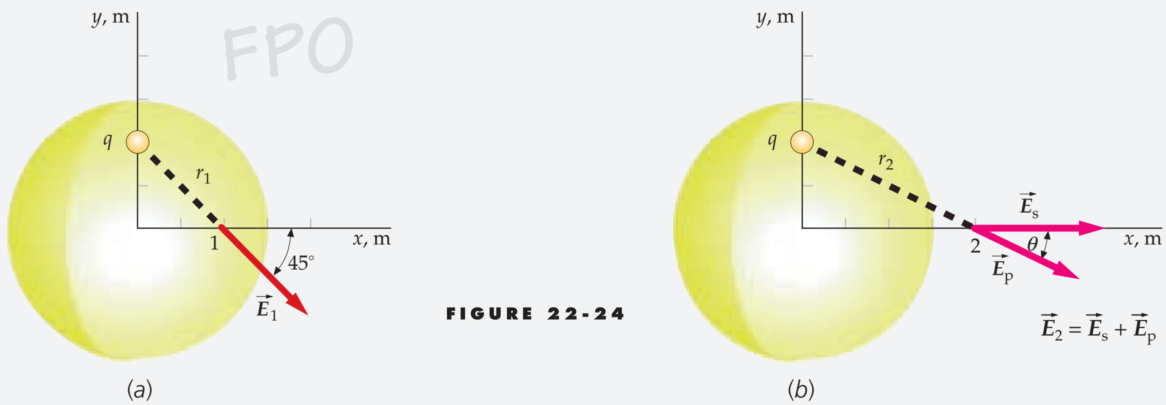


FIGURE 22-24

(a) 1. Inside the shell, $\vec{E}_1$ is due only to the point charge: $\vec{E}_1 = \dfrac{kq}{r_1^2}\hat{r}_1$

2. Calculate the square of the distance r_1: $r_1^2 = (2\text{ m})^2 + (2\text{ m})^2 = 8\text{ m}^2$

3. Use r_1^2 to calculate the magnitude of the field:
$$E_1 = \frac{kq}{r_1^2} = \frac{(8.99\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(250\times10^{-9}\text{ C})}{8\text{ m}^2} = 281\text{ N/C}$$

4. From Figure 22-24a, we can see that the field makes an angle of 45° with the x axis: $\theta_1 = 45°$

5. Express $\vec{E}_1$ in terms of its components:
$$\vec{E}_1 = E_{1x}\hat{i} + E_{1y}\hat{j} = E_1\cos 45°\hat{i} - E_1\sin 45°\hat{j}$$
$$= (281\text{ N/C})\cos 45°\hat{i} - (281\text{ N/C})\sin 45°\hat{j}$$
$$= \boxed{199\,(\hat{i} - \hat{j})\text{ N/C}}$$

(b) 1. Outside of its perimeter, the shell can be treated as a point charge at the origin, and the field due to the shell $\vec{E}_s$ is therefore along the x axis: $\vec{E}_s = \dfrac{kQ}{x_2^2}\hat{i}$

2. Calculate the total charge Q on the shell: $Q = \sigma 4\pi R^2 = (3\text{ nC/m}^2)4\pi(3\text{ m})^2 = 339\text{ nC}$

3. Use Q to calculate the field due to the shell:
$$E_s = \frac{kQ}{x_2^2} = \frac{(8.99\times10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(339\times10^{-9}\text{ C})}{(4\text{ m})^2} = 190\text{ N/C}$$

4. The field due to the point charge is: $\vec{E}_p = \dfrac{kq}{r_2^2}\hat{r}_2$

5. Calculate the square of the distance from the point charge q on the y axis to the field point at $x = 4$ m:

$$r_2^2 = (2\text{ m})^2 + (4\text{ m})^2 = 20\text{ m}^2$$

6. Calculate the magnitude of the field due to the point charge:

$$E_p = \frac{kq}{r_2^2} = \frac{(8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)(250 \times 10^{-9}\text{ C})}{20\text{ m}^2}$$
$$= 112\text{ N/C}$$

7. This field makes an angle θ with the x axis, where:

$$\tan\theta = \frac{2\text{ m}}{4\text{ m}} = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\frac{1}{2} = 26.6°$$

8. The x and y components of the net electric field are thus:

$$E_x = E_{px} + E_{sx} = E_p\cos\theta + E_s$$
$$= (112\text{ N/C})\cos 26.6° + 190\text{ N/C} = 290\text{ N/C}$$
$$E_y = E_{py} + E_{sy} = -E_p\sin\theta + 0$$
$$= -(112\text{ N/C})\sin 26.6° = -50.0\text{ N/C}$$
$$\vec{E} = \boxed{(290\hat{\imath} - 50.0\hat{\jmath})\text{N/C}}$$

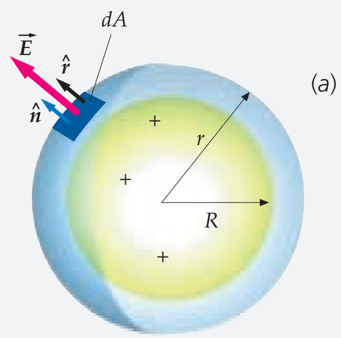
REMARKS Giving the x, y, and z components of a vector completely specifies the vector. In these cases, the z component is zero.

$\vec{E}$ Due to a Uniformly Charged Sphere

ELECTRIC FIELD DUE TO A CHARGED SOLID SPHERE **EXAMPLE 22-8**

Find the electric field (*a*) outside and (*b*) inside a uniformly charged solid sphere of radius R carrying a total charge Q that is uniformly distributed throughout the volume of the sphere with charge density $\rho = Q/V$, where $V = \frac{4}{3}\pi R^3$ is the volume of the sphere.

FIGURE 22-25



PICTURE THE PROBLEM By symmetry, the electric field must be radial. (*a*) To find E_r outside the charged sphere, we choose a spherical Gaussian surface of radius $r > R$ (Figure 22-25*a*). (*b*) To find E_r inside the charge we choose a spherical Gaussian surface of radius $r > R$ (Figure 22-25*b*). On each of these surfaces, E_r is constant. Gauss's law then relates E_r to the total charge inside the Gaussian surface.

(*a*) 1. (Outside) Draw a charged sphere of radius R and draw a spherical Gaussian surface with radius $r > R$:

2. Relate the flux through the Gaussian surface to the electric field E_r on it. At every point on this surface $\hat{n} = \hat{r}$ and E_r has the same value:

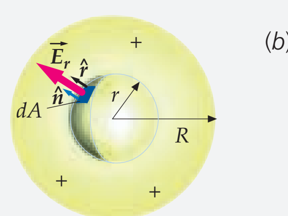
$$\phi_{\text{net}} = \vec{E}\cdot\hat{n}A = \vec{E}\cdot\hat{r}A = E_r 4\pi r^2$$

3. Apply Gauss's law to relate the field to the total charge inside the surface, which is Q:

$$E_r 4\pi r^2 = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

4. Solve for E_r:

$$E_r = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \quad r > R}$$



(*b*) 1. (Inside) Again draw the charged sphere of radius R. This time draw a spherical Gaussian surface with radius $r < R$:

2. Relate the flux through the Gaussian surface to the electric field E_r on it. At every point on this surface $\hat{n} = \hat{r}$ and E_r has the same value:

$$\phi_{\text{net}} = \vec{E}\cdot\hat{n}A = \vec{E}\cdot\hat{r}A = E_r 4\pi r^2$$

3. Apply Gauss's law to relate the field to the total charge inside the surface Q_{inside}:

$$E_r 4\pi r^2 = \frac{Q_{\text{inside}}}{\epsilon_0}$$

4. The total charge inside the surface is $\rho V'$, where $\rho = Q/V$, $V = \frac{4}{3}\pi R^3$ and $V' = \frac{4}{3}\pi r^3$. V is the volume of the solid sphere and V' is the volume inside the Gaussian surface:

$$Q_{\text{inside}} = \rho V' = \left(\frac{Q}{V}\right)V' = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{4}{3}\pi r^3\right) = Q\frac{r^3}{R^3}$$

5. Substitute this value for Q_{inside} and solve for E_r:

$$E_r 4\pi r^2 = \frac{Q_{\text{inside}}}{\epsilon_0} = \frac{1}{\epsilon_0} Q \frac{r^3}{R^3}$$

$$E_r = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r, \quad r \le R}$$

REMARKS Figure 22-26 shows E_r versus r for the charge distribution in this example. Inside a sphere of charge, E_r increases with r. Note that E_r is continuous at $r = R$. A uniformly charged sphere is sometimes used as a model to describe the electric field of an atomic nucleus.

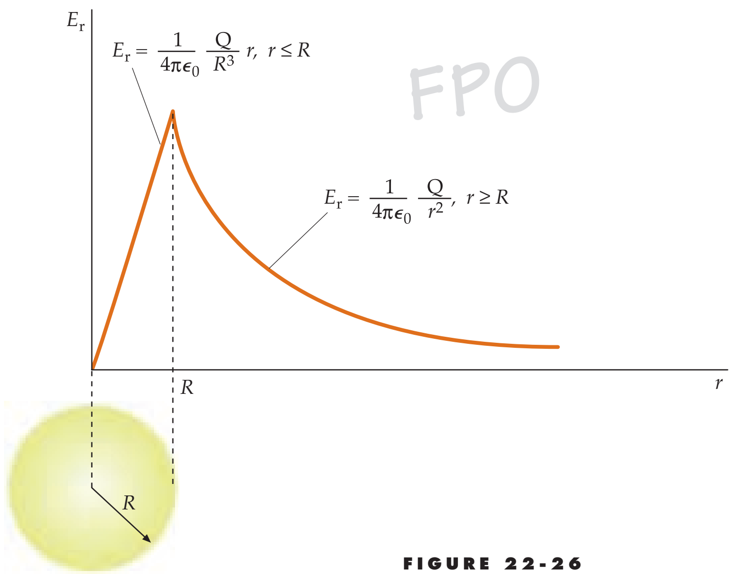


FIGURE 22-26

We see from Example 22-8 that the electric field a distance r from the center of a uniformly charged sphere of radius R is given by

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad r \ge R \tag{22-26a}$$

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r, \qquad r \le R \tag{22-26b}$$

where Q is the total charge of the sphere.

Cylindrical Symmetry

Consider a coaxial surface and charge distribution. A charge distribution has **cylindrical symmetry** if the views of it from all points on a cylindrical surface of infinite length are the same. To calculate the electric field due to cylindrically symmetric charge distributions, we use a cylindrical Gaussian surface. We illustrate this by calculating the electric field due to an infinitely long line charge of uniform linear charge density, a problem we have already solved using Coulomb's law.

Electric Field Due to Infinite Line Charge — EXAMPLE 22-9

Use Gauss's law to find the electric field everywhere due to an infinitely long line charge of uniform charge density λ.

PICTURE THE PROBLEM Because of the symmetry, we know the electric field is directed away if λ is positive (directly toward it if λ is negative), and we know the magnitude of the field depends only on the radial distance from the line charge. We therefore choose a soup-can shaped Gaussian surface coaxial with the line. This surface consists of three pieces, the two flat ends and the curved side. We calculate the outward flux of $\vec{E}$ through each piece and, using Gauss's law, relate the net outward flux to the charge density λ.

1. Sketch the wire and a coaxial soup-can shaped Gaussian surface (Figure 22-27) with length L and radius R. The closed surface consists of three pieces, the two flat ends and the curved side. At a randomly chosen point on each piece, draw the vectors $\vec{E}$ and $\hat{n}$. Because of the symmetry, we know that the direction of $\vec{E}$ is directly away from the line charge if λ is positive (directly toward it if λ is negative), and we know that the magnitude of E depends only on the radial distance from the line charge.

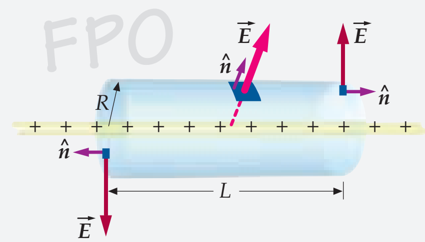


FIGURE 22-27

2. Calculate the outward flux through the curved piece of the Gaussian surface. At each point on the curved piece $\hat{R} = \hat{n}$, where $\hat{R}$ is the unit vector in the radial direction.

$$\phi_{\text{curved}} = \vec{E} \cdot \hat{n} A_{\text{curved}} = \vec{E} \cdot \hat{R} A_{\text{curved}} = E_R 2\pi R L$$

3. Calculate the outward flux through each of the flat ends of the Gaussian surface. On these pieces the direction of $\hat{n}$ is parallel with the line charge (and thus perpendicular to $\vec{E}$):

$$\phi_{\text{left}} = \vec{E} \cdot \hat{n} A_{\text{left}} = 0$$

$$\phi_{\text{right}} = \vec{E} \cdot \hat{n} A_{\text{right}} = 0$$

4. Apply Gauss's law to relate the field to the total charge inside the surface Q_{inside}. The net flux out of the Gaussian surface is the sum of the fluxes out of the three pieces of the surface, and Q_{inside} is the charge on a length L of the line charge:

$$\phi_{\text{net}} = \frac{Q_{\text{inside}}}{\epsilon_0}$$

$$E_R 2\pi R L = \frac{\lambda L}{\epsilon_0}$$

so

$$E_R = \boxed{\frac{1}{2\pi\,\epsilon_0}\frac{\lambda}{R}}$$

REMARKS Since $1/(2\pi\epsilon_0) = 2k$, the field is $2k\lambda/R$, the same as Equation 22-9.

It is important to realize that although Gauss's law holds for any surface surrounding any charge distribution, it is very useful for calculating the electric fields of charge distributions that are highly symmetric. It is also useful doing calculations involving conductors in electrostatic equilibrium, as we shall see in Section 22.5. In the calculation of Example 22-9, we needed to assume that the field point was very far from the ends of the line charge so that E_n would be constant everywhere on the cylindrical Gaussian surface. (This is equivalent to assuming that, at the distance R from the line, the line charge appears to be infinitely long.) If we are near the end of a finite line charge, we cannot assume that $\vec{E}$ is perpendicular to the curved surface of the soup can, or that E_n is constant everywhere on it, so we cannot use Gauss's law to calculate the electric field.

22-4 Discontinuity of E_n

We have seen that the electric field for an infinite plane of charge and a thin spherical shell of charge is discontinuous by the amount σ/ϵ_0 on either side of a surface carrying charge density σ. We now show that this is a general result for the component of the electric field that is perpendicular to a surface carrying a charge density of σ.

Figure 22-28 shows an arbitrary surface carrying a surface charge density σ. The surface is arbitrary in that it is arbitrarily curved, although it does not have any sharp folds, and σ may vary continuously

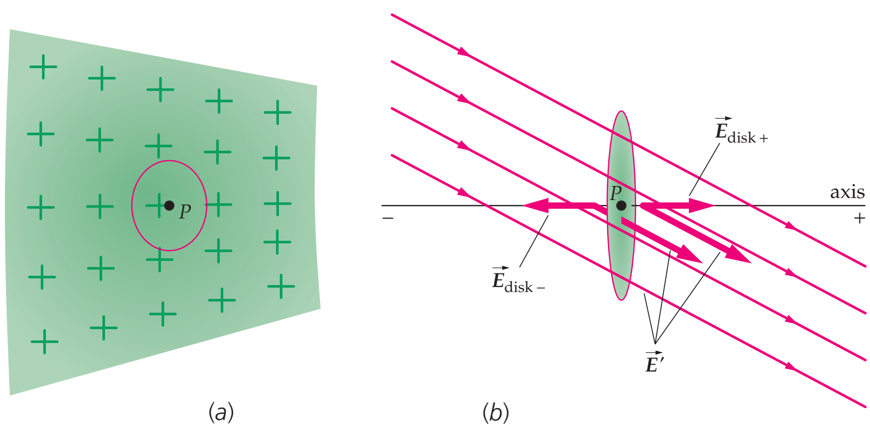


FIGURE 22-28 (*a*) A surface carrying surface-charge. (*b*) The electric field $\vec{E}_{\text{disk}}$ due to the charge on a circular disk, plus the electric field $\vec{E}'$ due to all other charges. The right side of the disk is the + side, the left side the − side.

on the surface from place to place. We consider electric field $\vec{E}$ in the vicinity of a point P on the surface as the superposition of electric field $\vec{E}_{\text{disk}}$, due just to the charge on a small disk centered at point P, and $\vec{E}$ due to all other charges in the universe. Thus,

$$\vec{E} = \vec{E}_{\text{disk}} + \vec{E}' \qquad \text{22-27}$$

The disk is small enough that it may be considered both flat and uniformly charged. On the axis of the disk, the electric field $\vec{E}_{\text{disk}}$ is given by Equation 22-11. At points on the axis very close to the disk, the magnitude of this field is given by $E_{\text{disk}} = |\sigma|/(2\,\epsilon_0)$ and its direction is away from the disk if σ is positive, and toward it if σ is negative. The magnitude and direction of the electric field $\vec{E}'$ is unknown. In the vicinity of point P, however, this field is continuous. Thus, at points on the axis of the disk and very close to it, $\vec{E}'$ is essentially uniform.

The axis of the disk is normal to the surface, so vector components along this axis can be referred to as normal components. The normal components of the vectors in Equation 22-27 are related by $E_n = E_{\text{disk}\,n} + E'_n$. If we refer one side of the surface as the + side, and the other side the − side, then $E_{n+} = \dfrac{\sigma}{2\,\epsilon_0} + E'_{n+}$ and $E_{n-} = -\dfrac{\sigma}{2\,\epsilon_0} + E'_{n+}$. Thus, E_n changes discontinuously from one side of the surface to the other. That is:

$$\Delta E_n = E_{n+} - E_{n-} = \frac{\sigma}{2\,\epsilon_0} - \left(-\frac{\sigma}{2\,\epsilon_0}\right) = \frac{\sigma}{\epsilon_0} \qquad \text{22-28}$$

DISCONTINUITY OF E_n AT A SURFACE CHARGE

where we have made use of the fact that near the disk $E'_{n+} = E'_{n-}$ (since $\vec{E}'$ is continuous and uniform).

Note that the discontinuity of E_n occurs at a finite disk of charge, an infinite plane of charge (refer to Figure 22-10), and a thin spherical shell of charge (see Figure 22-23). However, it does not occur at the perimeter of a solid sphere of charge (see Figure 22-26). The electric field is discontinuous at any location with an infinite volume-charge density. These include locations with a finite point charge, locations with a finite line-charge density, and locations with a finite surface-charge density. At all locations with a finite surface-charge density, the normal component of the electric field is discontinuous—in accord with Equation 22-28.

22-5 Charge and Field at Conductor Surfaces

A conductor contains an enormous amount of mobile charge that can move freely within the conductor. If there is an electric field within a conductor, there will be a net force on this charge causing a momentary electric current (electric currents are discussed in Chapter 25). However, unless there is a source of energy to maintain this current, the free charge in a conductor will merely redistribute itself to create an electric field that cancels the external field within the conductor. The conductor is then said to be in **electrostatic equilibrium.** Thus, in electrostatic equilibrium, the electric field inside a conductor is zero everywhere. The time taken to reach equilibrium depends on the conductor. For copper and other metal conductors, the time is so small that in most cases electrostatic equilibrium is reached in a few nanoseconds.

We can use Gauss's law to show that any net electric charge on a conductor resides on the surface of the conductor. Consider a Gaussian surface completely inside the material of a conductor in electrostatic equilibrium (Figure 22-29). The size and shape of the Gaussian surface doesn't matter, as long as the entire surface is within the material of the conductor. The electric field is zero everywhere on the Gaussian surface because the surface is completely within the conductor where the field is everywhere zero. The net flux of the electric field through the surface must therefore be zero, and, by Gauss's law, the net charge inside the surface must be zero. Thus, there can be no net charge inside any surface lying completely within the material of the conductor. If a conductor carries a net charge, it must reside on the conductor's surface. At the surface of a conductor in electrostatic equilibrium, $\vec{E}$ must be perpendicular to the surface. We conclude this by reasoning that if the electric field had a tangential component at the surface, the free charge would be accelerated tangential to the surface until electrostatic equilibrium was reestablished.

Since E_n is discontinuous at any charged surface by the amount σ/ϵ_0, and since $\vec{E}$ is zero inside the material of a conductor, the field just outside the surface of a conductor is given by

$$E_n = \frac{\sigma}{\epsilon_0} \qquad 22\text{-}29$$

E_n JUST OUTSIDE THE SURFACE OF A CONDUCTOR

This result is exactly twice the field produced by a uniform disk of charge. We can understand this result from Figure 22-30. The charge on the conductor consists of two parts: (1) the charge near point P and (2) all the rest of the charge. The charge near point P looks like a small, uniformly charged circular disk centered at P that produces a field near P of magnitude $\sigma/(2\,\epsilon_0)$ just inside and just outside the conductor. The rest of the charges in the universe must produce a field of magnitude $\sigma/(2\,\epsilon_0)$ that exactly cancels the field inside the conductor. This field due to the rest of the charge adds to the field due to the small charged disk just outside the conductor to give a total field of σ/ϵ_0.

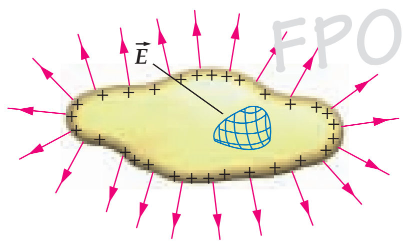


FIGURE 22-29 A Gaussian surface completely within the material of a conductor. Since the electric field is zero inside a conductor in electrostatic equilibrium, the net flux through this surface must also be zero. Therefore, the net charge density within the material of a conductor must be zero.

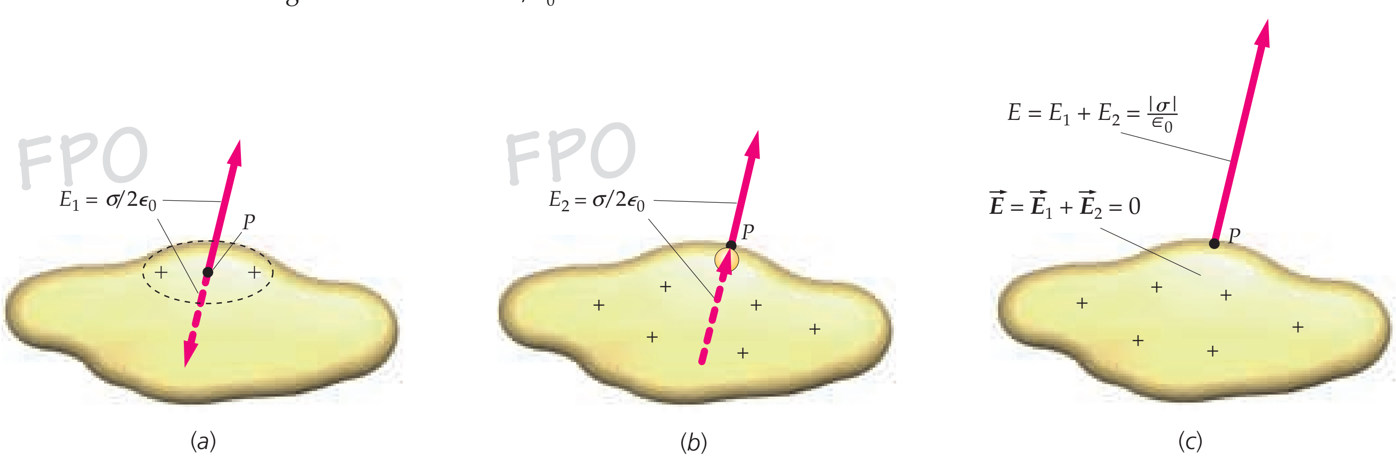


FIGURE 22-30 An arbitrarily shaped conductor carrying a charge on its surface. (*a*) The charge in the vicinity of point P near the surface looks like a small uniformly charged circular disk centered at P, giving an electric field of magnitude $\sigma/(2\epsilon_0)$ pointing away from the surface both inside and outside the surface. Inside the conductor, this field points down from point P. (*b*) Since the net field inside the conductor is zero, the rest of the charges in the universe must produce a field of magnitude $\sigma/(2\epsilon_0)$ in the outward direction. The field due to this charge is the same just inside the surface as it is just outside the surface. (*c*) Inside the surface, the fields shown in (*a*) and (*b*) cancel, but outside at point P they add to give $E_n = \sigma/\epsilon^0$.

The Charge of the Earth **EXAMPLE 22-10**

While watching a science show on the atmosphere, you find out that on average the electric field of the Earth is about 100 N/C directed vertically downwards. Given that you have been studying electric fields in your physics class, you wonder if you can determine what the total charge on the Earth's surface is.

PICTURE THE PROBLEM The earth is a conductor, so any charge it carries resides on the surface of the earth. The surface charge density σ is related to the normal component of the electric field E_n by Equation 22-29. The total charge Q equals the charge density σ times the surface area A.

1. The surface charge density σ is related to the normal component of the electric field E_n by Equation 22-29: $E_n = \dfrac{\sigma}{\epsilon_0}$

2. On the surface of the earth $\hat{n}$ is upward and $\vec{E}$ is downward, so E_n is negative: $E_n = \vec{E} \cdot \hat{n} = E \times 1 \times \cos 180° = -E = -100\text{ n/C}$

3. The charge Q is the charge per unit area. Combine this with the step 1 and 2 results to obtain an expression for Q: $Q = \sigma A = \epsilon_0 E_n A = -\epsilon_0 EA$

4. The surface area of a sphere of radius r is given by $A = 4\pi r^2$. $Q = -\epsilon_0 EA = -\epsilon_0 E 4\pi R_E^2 = -4\pi\epsilon_0 E R_E^2$

5. The radius of the earth is 6.38×10^{-6} m:

$$Q = -4\pi\epsilon_0 E R_E^2$$
$$= -4\pi(8.85 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2)(100\text{ N/C})(6.38 \times 10^6\text{ m})^2$$
$$= \boxed{-4.53 \times 10^5\text{ C}}$$

Figure 22-31 shows a positive point charge q at the center of a spherical cavity inside a spherical conductor. Since the net charge must be zero within any Gaussian surface drawn within the conductor, there must be a negative charge $-q$ induced in the inside surface. In Figure 22-32, the point charge has been moved so that it is no longer at the center of the cavity. The field lines in the cavity are altered, and the surface charge density of the induced negative charge on the inner surface is no longer uniform. However, the positive surface charge density on the outside surface is not disturbed—it is still uniform—because it is electrically shielded from the cavity by the conducting material.

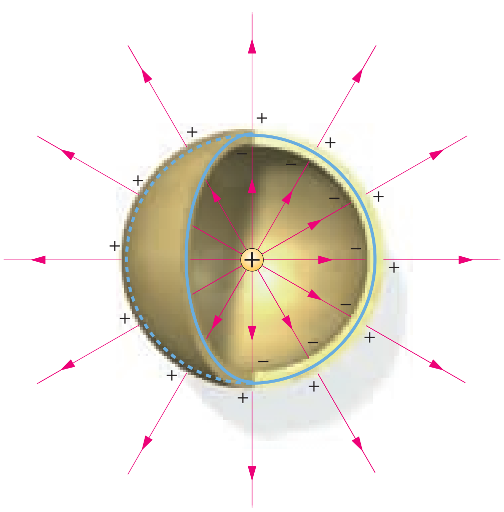


FIGURE 22-31 A point charge q in the cavity at the center of a thick spherical conducting shell. Since the net charge within the Gaussian surface (indicated in blue) must be zero, a surface charge $-q$ is induced on the inner surface of the shell, and since the conductor is neutral, an equal but opposite charge $+q$ is induced on the outer surface. Electric field lines begin on the point charge and end on the inner surface. Field lines begin again on the outer surface.

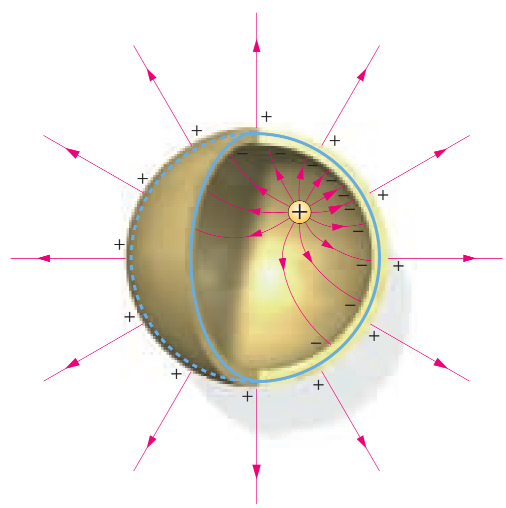


FIGURE 22-32 The same conductor as in Figure 22-31 with the point charge moved away from the center of the sphere. The charge on the outer surface and the electric field lines outside the sphere are not affected.

Electric field lines for an oppositely charged cylinder and plate, shown by bits of fine thread suspended in oil. Note that the field lines are perpendicular to the conductors and that there are no lines inside the cylinder.

ELECTRIC FIELD ON TWO FACES OF A DISK — EXAMPLE 22-11

An infinite, nonconducting, uniformly charged plane is located in the $x = -a$ plane, and a second such plane is located in the $x = +a$ plane (Figure 22-33*a*). The plane at $x = -a$ carries a positive charge density whereas the plane at $x = +a$ carries a negative charge density of the same magnitude. The electric field due to the charges on both planes is $\vec{E}_{\text{applied}} = (450\ \text{kN/C})\hat{i}$ in the region between them. A thin, uncharged 2-m radius conducting disk is placed in the $x = 0$ plane and centered at the origin (Figure 22-33*b*). (*a*) Find the charge density on each face of the disk. Also, find the electric field just outside the disk at each face. (Assume that any charge on either face is uniformly distributed.) (*b*) A net charge of 96 μC is placed on the disk. Find the new charge density on each face and the electric field just outside each face but far from the edges of the sheet.

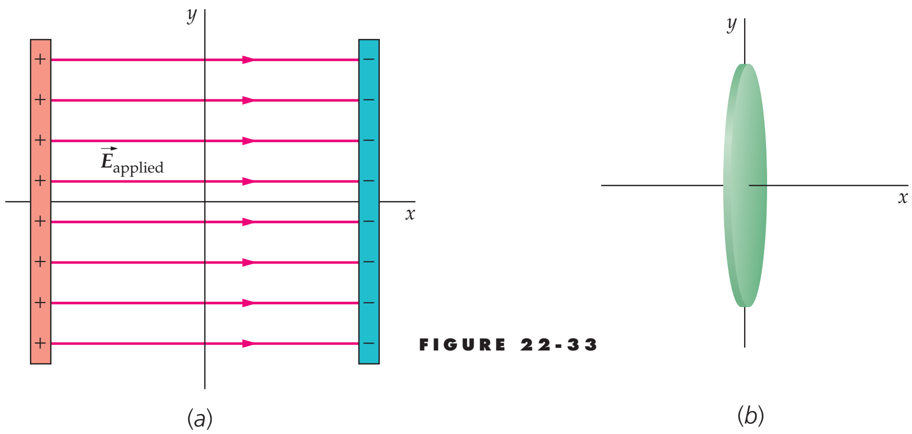


FIGURE 22-33

PICTURE THE PROBLEM (*a*) We find the charge density by using the fact that the total charge on the disk is zero and that there is no electric field inside the conducting material of the disk. The surface charges on the disk must produce an electric field inside it that exactly cancels $\vec{E}_{\text{applied}}$. (*b*) The additional charge of 96 μC must be distributed so that the electric field inside the conducting disk remains zero.

(*a*) 1. Let σ_a and σ_L be the charge densities on the right and left faces on the conducting sheet, respectively. Since the disk is uncharged, these densities must add to zero.

$$\sigma_R + \sigma_L = 0$$

so

$$\sigma_L = -\sigma_R$$

2. Inside the conducting sheet the electric field due to the charges on its surface must cancel $\vec{E}_{\text{applied}}$. Let $\vec{E}_R$ and $\vec{E}_L$ be the electric field due to the charges on the right and left faces, respectively.

$$\vec{E}_R + \vec{E}_L + \vec{E}_{\text{applied}} = 0$$

3. Using Equations 22-13*a* and *b* we can express the electric field due to the charge on each surface of the disk by the corresponding surface charge density. The field due to a disk of surface charge σ next to the disk is given by $[\sigma/(2\,\epsilon_0)]\,\hat{u}$, where $\hat{u}$ is a unit vector directed away from the surface charge.

$$\vec{E}_R + \vec{E}_L + \vec{E}_{\text{applied}} = 0$$

$$\frac{\sigma_R}{2\,\epsilon_0}(-\hat{i}) + \frac{\sigma_L}{2\,\epsilon_0}\hat{i} + \vec{E}_{\text{applied}} = 0$$

4. Substituting $-\sigma_R$ for σ_L and solving for the surface charge densities gives:

$$\frac{\sigma_R}{2\,\epsilon_0}(-\hat{i}) + \frac{-\sigma_R}{2\,\epsilon_0}\hat{i} + \vec{E}_{\text{applied}} = 0$$

$$-\frac{\sigma_R}{\epsilon_0}\hat{i} + \vec{E}_{\text{applied}} = 0$$

$$\sigma_R\hat{\boldsymbol{i}} = \epsilon_0 \vec{E}_{\text{applied}}$$
$$= (8.85 \times 10^{-12}\ \text{C}^2/\text{N·m}^2)(450\ \text{kN/C})\hat{\boldsymbol{i}}$$
$$\sigma_R = 3.98 \times 10^{-6}\ \text{C/m}^2 = \boxed{3.98\ \mu\text{C/m}^2}$$
$$\sigma_L = -\sigma_R = \boxed{-3.98\ \mu\text{C/m}^2}$$

5. Use Equation 22-29 ($E_n = \sigma/\epsilon_0$) to relate the electric field just outside a conductor to the surface charge density on it. Just outside the right side of the disk $\hat{\boldsymbol{n}} = \hat{\boldsymbol{i}}$, and just outside the left side $\hat{\boldsymbol{n}} = -\hat{\boldsymbol{i}}$:

$$E_{Rn} = \frac{\sigma_R}{\epsilon_0} = \frac{3.98\ \text{kC/m}^2}{8.85 \times 10^{-12}\ \text{C}^2/\text{N·m}^2}$$
$$= 450\ \text{kN/C}$$
$$\vec{E}_R = E_{Rn}\hat{\boldsymbol{n}} = E_{Rn}\hat{\boldsymbol{i}}\ \boxed{450\ \text{kN/C}\hat{\boldsymbol{i}}}$$
$$E_{Ln} = \frac{\sigma_L}{\epsilon_0} = \frac{-3.98\ \text{kC/m}^2}{8.85 \times 10^{-12}\ \text{C}^2/\text{N·m}^2}$$
$$\vec{E}_L = E_{Ln}\hat{\boldsymbol{n}} = E_{Ln}(-\hat{\boldsymbol{i}}) = \boxed{450\ \text{kN/C}\hat{\boldsymbol{i}}}$$

(*b*) 1. The sum of the charges on the two faces of the disk must equal the net charge on the disk.

$$Q_R + Q_L = Q_{\text{net}}$$
$$\sigma_R A + \sigma_L A = Q_{\text{net}}$$

or

$$\sigma_L = \frac{Q_{\text{net}}}{A} - \sigma_R$$

2. Substitute for σ_L in the Part (*a*), step 2 result and solve for the surface charge densities:

$$\frac{\sigma_R}{2\,\epsilon_0}(-\hat{\boldsymbol{i}}) + \frac{(Q_{\text{net}}/A) - \sigma_R}{2\,\epsilon_0}\hat{\boldsymbol{i}} + \vec{E}_{\text{applied}} = 0$$
$$\frac{(Q_{\text{net}}/A) - 2\sigma_R}{2\,\epsilon_0}\hat{\boldsymbol{i}} + \vec{E}_{\text{applied}} = 0$$
$$\sigma_R\hat{\boldsymbol{i}} = \epsilon_0 \vec{E}_{\text{applied}} + \frac{Q_{\text{net}}}{2A}\hat{\boldsymbol{i}} = \epsilon_0\,(450\ \text{kN/C})\hat{\boldsymbol{i}} + \frac{Q_{\text{net}}}{2A}\hat{\boldsymbol{i}}$$
$$\sigma_R = (8.85 \times 10^{-12}\ \text{C}^2/\text{N·m}^2)(450\ \text{kN/C}) + \frac{Q_{\text{net}}}{2A}$$
$$= 3.98\ \mu\text{C/m}^2 + \frac{96\ \mu\text{C}}{2\pi(1\ \text{m})^2} = \boxed{19.3\ \mu\text{C/m}^2}$$
$$\sigma_L = \frac{Q_{\text{net}}}{A} - \sigma_R = \frac{Q_{\text{net}}}{A} - \left(\epsilon_0\,(450\ \text{kN/C}) + \frac{Q_{\text{net}}}{2A}\right)$$
$$= -\epsilon_0\,(450\ \text{kN/C}) + \frac{Q_{\text{net}}}{2A}$$
$$= -3.98\ \mu\text{C/m}^2 + \frac{96\ \mu\text{C}}{2\pi(1\ \text{m})^2} = \boxed{11.3\ \mu\text{C/m}^2}$$

3. Using Equation 22-29 ($E_n = \epsilon_0\sigma$), relate the electric field just outside a conductor to the surface charge density on it.

$$E_{Rn} = \frac{\sigma_R}{\epsilon_0} = \frac{19.3\ \mu\text{C/m}^2}{8.85 \times 10^{12}\ \text{C}^2/\text{N·m}^2}$$
$$= 2.17 \times 10^6\ \text{N/C}$$
$$\vec{E}_R = E_{Rn}\hat{\boldsymbol{n}} = E_{Rn}\hat{\boldsymbol{i}} = \boxed{+217\ \text{MN/C}\hat{\boldsymbol{i}}}$$
$$E_{Ln} = \frac{\sigma_L}{\epsilon_0} = \frac{11.3\ \mu\text{C/m}^2}{8.85 \times 10^{12}\ \text{C}^2/\text{N·m}^2}$$
$$\vec{E}_L = E_{Ln}\hat{\boldsymbol{n}} = E_{Ln}(-\hat{\boldsymbol{i}}) = \boxed{-1.28\ \text{MN/C}\hat{\boldsymbol{i}}}$$

REMARKS The charge added to the disk was distributed equally, half on one side and half on the other. The electric field inside the disk due to this added charge is exactly zero. On each side of a real charged conducting thin disk the magnitude of the charge density is greatest near the edge of the disk.

EXERCISE The electric field just outside the surface of a certain conductor points away from the conductor and has a magnitude of 2000 N/C. What is the surface charge density on the surface of the conductor? (*Answer* 17.7 nC/m^2)

*22-6 Derivation of Gauss's Law From Coulomb's Law

Gauss's law can be derived mathematically using the concept of the **solid angle.** Consider an area element ΔA on a spherical surface. The solid angle $\Delta\Omega$ subtended by ΔA at the center of the sphere is defined to be

$$\Delta\Omega = \frac{\Delta A}{r^2}$$

where r is the radius of the sphere. Since ΔA and r^2 both have dimensions of length squared, the solid angle is dimensionless. The SI unit of the solid angle is the **steradian** (sr). Since the total area of a sphere is $4\pi r^2$, the total solid angle subtended by a sphere is

$$\frac{4\pi r^2}{r^2} = 4\pi \text{ steradians}$$

There is a close analogy between the solid angle and the ordinary plane angle $\Delta\theta$, which is defined to be the ratio of an element of arc length of a circle Δs to the radius of the circle:

$$\Delta\theta = \frac{\Delta s}{r} \text{ radians}$$

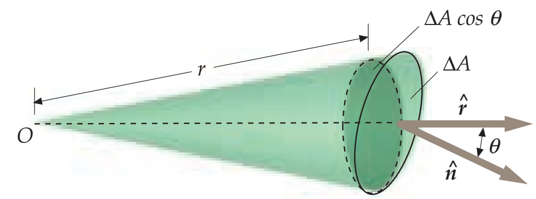


FIGURE 22-34 **An area element ΔA whose normal is not parallel to the radial line from O to the center of the element. The solid angle subtended by this element at O is defined to be $(\Delta A \cos\theta)/r^2$.**

The total plane angle subtended by a circle is 2π radians.

In Figure 22-34, the area element ΔA is not perpendicular to the radial lines from point O. The unit vector $\hat{n}$ normal to the area element makes an angle θ with the radial unit vector $\hat{r}$. In this case, the solid angle subtended by ΔA at point O is

$$\Delta\Omega = \frac{\Delta A\, \hat{n}\cdot\hat{r}}{r^2} = \frac{\Delta A \cos\theta}{r^2} \qquad \text{22-30}$$

Figure 22-35 shows a point charge q surrounded by a surface S of arbitrary shape. To calculate the flux of $\vec{E}$ through this surface, we want to find $\vec{E}\cdot\hat{n}\Delta A$ for each element of area on the surface and sum over the entire surface. The electric field at the area element shown is given by

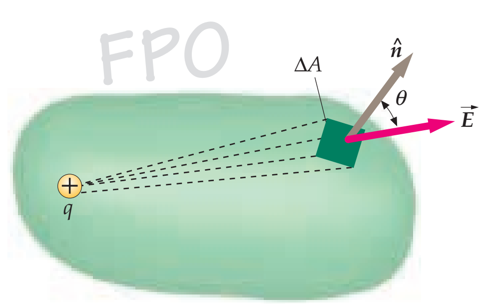


FIGURE 22-35 **A point charge enclosed by an arbitrary surface S. The flux through an area element ΔA is proportional to the solid angle subtended by the area element at the charge. The net flux through the surface, found by summing over all the area elements, is proportional to the total solid angle 4π at the charge, which is independent of the shape of the surface.**

$$\vec{E} = \frac{kq}{r^2}\hat{r}$$

so the flux through the element is

$$\Delta\phi = \vec{E} \cdot \hat{n}\,\Delta A = \frac{kq}{r^2}\hat{r} \cdot \hat{n}\,\Delta A = kq\,\Delta\Omega$$

The solid angle $\Delta\Omega$ is the same as that subtended by the corresponding area element of a spherical surface of any radius. The sum of the fluxes through the entire surface is kq times the total solid angle subtended by the closed surface, which is 4π steradians:

$$\phi_{\text{net}} = \oint_S \vec{E} \cdot \hat{n}\,dA = kq \oint d\Omega = kq4\pi = 4\pi kq = \frac{q}{\epsilon_0} \qquad \text{22-31}$$

which is Gauss's law.

SUMMARY

1. Gauss's law is a fundamental law of physics that is equivalent to Coulomb's law for static charges.
2. For highly symmetric charge distributions, Gauss's law can be used to calculate the electric field.

Topic	Relevant Equations and Remarks	
1. Electric Field for a Continuous Charge Distribution	$\vec{E} = \int_V \frac{k\,dq}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\int_V \frac{dq}{r^2}\hat{r}$ (Coulomb's law) where $dq = \rho\,dV$ for a charge distributed throughout a volume, $dq = \sigma\,dA$ for a charge distributed on a surface, and $dq = \lambda\,dL$ for a charge distributed along a line.	**22-4**
2. Electric Flux	$\phi = \lim_{\Delta A_i \to 0} \sum_i \vec{E}_i \cdot \hat{n}_i\,\Delta A_i = \int_S \vec{E} \cdot \hat{n}dA$	**22-16**
3. Gauss's Law	$\phi_{\text{net}} = \int_S E_n\,dA = 4\pi k\,Q_{\text{inside}} = \frac{Q_{\text{inside}}}{\epsilon_0}$ The net outward flux through a closed surface equals $4\pi k$ times the net charge within the surface.	**22-19**
4. Coulomb Constant k and Permittivity of Free Space ϵ_0	$k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2$ $\epsilon_0 = \frac{1}{4\pi k} = 8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$	**22-23**

5. Coulomb's Law and Gauss's Law	$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$	**22-21**
	$\phi_{net} = \oint_S E_n\,dA = \frac{Q_{inside}}{\epsilon_0}$	**22-22**
6. Discontinuity of E_n	At a surface carrying a surface charge density σ, the component of the electric field perpendicular to the surface is discontinuous by σ/ϵ_0.	
	$E_{n+} - E_{n-} = \frac{\sigma}{\epsilon_0}$	**22-28**
7. Charge on a Conductor	In electrostatic equilibrium, the net electric charge on a conductor resides on the surface of the conductor.	
8. $\vec{E}$ Just Outside a Conductor	The resultant electric field just outside the surface of a conductor is perpendicular to the surface and has the magnitude σ/ϵ_0, where σ is the local surface charge density at that point on the conductor:	
	$E_n = \frac{\sigma}{\epsilon_0}$	**22-29**
	The force per unit area exerted on the charge on the surface of a conductor by all the other charges is called the electrostatic stress.	
9. Electric Fields for Various Uniform Charge Distributions		
Of a line charge	$E_y = \frac{k\lambda}{y}(\sin\theta_2 - \sin\theta_1)$; $E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1)$	**22-8**
Of a line charge of infinite length	$E_R = 2k\frac{\lambda}{R} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{R}$	**22-9**
On the axis of a charged ring	$E_x = \frac{kQx}{(x^2 + a^2)^{3/2}}$	**22-10**
On the axis of a charged disk	$E_x = \frac{\sigma}{2\epsilon_0}\left(1 - \frac{1}{\sqrt{1 + \frac{R^2}{x^2}}}\right), \qquad x > 0$	**22-11**
Of a charged plane	$E_x = \frac{\sigma}{2\epsilon_0}, \qquad x > 0$	**22-24**
Of a charged spherical shell	$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad r > R$	**22-25*a***
	$E_r = 0, \qquad r < R$	**22-25*b***
Of a charged solid sphere	$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}, \qquad r \geq R$	**22-26*a***
	$E_r = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}r, \qquad r \leq R$	**22-26*b***

PROBLEMS

- • Single-concept, single-step, relatively easy
- •• Intermediate-level, may require synthesis of concepts
- ••• Challenging
- SSM Solution is in the *Student Solutions Manual*
- iSOLVE Problems available on iSOLVE online homework service
- iSOLVE✓ These "Checkpoint" online homework service problems ask students additional questions about their confidence level, and how they arrived at their answer.

In a few problems, you are given more data than you actually need; in a few other problems, you are required to supply data from your general knowledge, outside sources, or informed estimates.

Conceptual Problems

1 •• SSM True or false:

(*a*) Gauss's law holds only for symmetric charge distributions.
(*b*) The result that $E = 0$ inside a conductor can be derived from Gauss's law.

2 •• What information, in addition to the total charge inside a surface, is needed to use Gauss's law to find the electric field?

3 ••• Is the electric field E in Gauss's law only that part of the electric field due to the charge inside a surface, or is it the total electric field due to all charges both inside and outside the surface?

4 •• Explain why the electric field increases with r rather than decreasing as $1/r^2$ as one moves out from the center inside a spherical charge distribution of constant volume charge density.

5 • SSM True or false:

(*a*) If there is no charge in a region of space, the electric field on a surface surrounding the region must be zero everywhere.
(*b*) The electric field inside a uniformly charged spherical shell is zero.
(*c*) In electrostatic equilibrium, the electric field inside a conductor is zero.
(*d*) If the net charge on a conductor is zero, the charge density must be zero at every point on the surface of the conductor.

6 • If the electric field E is zero everywhere on a closed surface, is the net flux through the surface necessarily zero? What, then, is the net charge inside the surface?

7 • A point charge $-Q$ is at the center of a spherical conducting shell of inner radius R_1 and outer radius R_2, as shown in Figure 22-36. The charge on the inner surface of the shell is (*a*) $+Q$. (*b*) zero. (*c*) $-Q$. (*d*) dependent on the total charge carried by the shell.

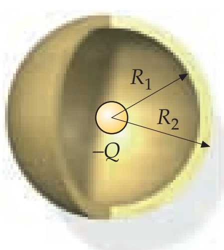


FIGURE 22-36 Problem 7

8 • For the configuration of Figure 22-36, the charge on the outer surface of the shell is (*a*) $+Q$. (*b*) zero. (*c*) $-Q$. (*d*) dependent on the total charge carried by the shell.

9 •• SSM Suppose that the total charge on the conducting shell of Figure 22-36 is zero. It follows that the electric field for $r < R_1$ and $r > R_2$ points

(*a*) away from the center of the shell in both regions.
(*b*) toward the center of the shell in both regions.
(*c*) toward the center of the shell for $r < R_1$ and is zero for $r > R_2$.
(*d*) away from the center of the shell for $r < R_1$ and is zero for $r > R_2$.

10 •• SSM If the conducting shell in Figure 22-36 is grounded, which of the following statements is then correct?

(*a*) The charge on the inner surface of the shell is $+Q$ and that on the outer surface is $-Q$.
(*b*) The charge on the inner surface of the shell is $+Q$ and that on the outer surface is zero.
(*c*) The charge on both surfaces of the shell is $+Q$.
(*d*) The charge on both surfaces of the shell is zero.

11 •• For the configuration described in Problem 10, in which the conducting shell is grounded, the electric field for $r < R_1$ and $r > R_2$ points

(*a*) away from the center of the shell in both regions.
(*b*) toward the center of the shell in both regions.
(*c*) toward the center of the shell for $r < R_1$ and is zero for $r > R_2$.
(*d*) toward the center of the shell for $r < R_1$ and is zero for $r > R_1$.

12 •• If the net flux through a closed surface is zero, does it follow that the electric field E is zero everywhere on the surface? Does it follow that the net charge inside the surface is zero?

13 •• True or false: The electric field is discontinuous at all points at which the charge density is discontinuous.

Estimation and Approximation

14 •• SSM Given that the maximum field sustainable in air without electrical discharge is approximately 3×10^6 V/m, estimate the total charge of a thundercloud. Make any assumptions that seem reasonable.

15 •• If you rub a rubber balloon against dry hair, the resulting static charge will be enough to make the hair stand on end. Estimate the surface charge density on the balloon and its electric field.

16 • A disk of radius 2.5 cm carries a uniform surface charge density of 3.6 $\mu C/m^2$. Using reasonable approximations, find the electric field on the axis at distances of (*a*) 0.01 cm, (*b*) 0.04 cm, (*c*) 5 m, and (*d*) 5 cm.

Calculating $\vec{E}$ From Coulomb's Law

17 • SSM A uniform line charge of linear charge density $\lambda = 3.5$ nC/m extends from $x = 0$ to $x = 5$ m. (*a*) What is the total charge? Find the electric field on the x axis at (*b*) $x = 6$ m, (*c*) $x = 9$ m, and (*d*) $x = 250$ m. (*e*) Find the field at $x = 250$ m, using the approximation that the charge is a point charge at the origin, and compare your result with that for the exact calculation in Part (*d*).

18 • Two infinite vertical planes of charge are parallel to each other and are separated by a distance $d = 4$ m. Find the electric field to the left of the planes, to the right of the planes, and between the planes (*a*) when each plane has a uniform surface charge density $\sigma = +3\ \mu C/m^2$ and (*b*) when the left plane has a uniform surface charge density $\sigma = +3\ \mu C/m^2$ and that of the right plane is $\sigma = -3\ \mu C/m^2$. Draw the electric field lines for each case.

19 • A 2.75-μC charge is uniformly distributed on a ring of radius 8.5 cm. Find the electric field on the axis at (*a*) 1.2 cm, (*b*) 3.6 cm, and (*c*) 4.0 m from the center of the ring. (*d*) Find the field at 4.0 m using the approximation that the ring is a point charge at the origin, and compare your results with that for Part (*c*).

20 • For the disk charge of Problem 16, calculate exactly the electric field on the axis at distances of (*a*) 0.04 cm and (*b*) 5 m, and compare your results with those for Parts (*b*) and (*c*) of Problem 16.

21 • A uniform line charge extends from $x = -2.5$ cm to $x = +2.5$ cm and has a linear charge density of $\lambda = 6.0$ nC/m. (*a*) Find the total charge. Find the electric field on the y axis at (*b*) $y = 4$ cm, (*c*) $y = 12$ cm, and (*d*) $y = 4.5$ m. (*e*) Find the field at $y = 4.5$ m, assuming the charge to be a point charge, and compare your result with that for Part (*d*).

22 • A disk of radius a lies in the yz plane with its axis along the x axis and carries a uniform surface charge density σ. Find the value of x for which $E_x = \frac{1}{2}\sigma/2\epsilon_0$.

23 • A ring of radius a with its center at the origin and its axis along the x axis carries a total charge Q. Find E_x at (*a*) $x = 0.2a$, (*b*) $x = 0.5a$, (*c*) $x = 0.7a$, (*d*) $x = a$, and (*e*) $x = 2a$. (*f*) Use your results to plot E_x versus x for both positive and negative values of x.

24 • Repeat Problem 23 for a disk of uniform surface charge density σ.

25 •• SSM (*a*) Using a spreadsheet program or graphing calculator, make a graph of the electric field on the axis of a disk of radius r = 30 cm carrying a surface charge density $\sigma = 0.5$ nC/m^2. (*b*) Compare the field to the approximation E = $2\pi k\sigma$. At what distance does the approximation differ from the exact solution by 10 percent?

26 •• Show that E_x on the axis of a ring charge of radius a has its maximum and minimum values at $x = +a/\sqrt{2}$ and $x = -a/\sqrt{2}$. Sketch E_x versus x for both positive and negative values of x.

27 •• A line charge of uniform linear charge density λ lies along the x axis from $x = x_1$ to $x = x_2$ where $x_1 < x_2$. Show the x component of the electric field at a point on the y axis is given by

$$E_x = \frac{k\lambda}{y}(\cos\theta_2 - \cos\theta_1)$$

where ($\theta_1 = \tan^{-1}(x_1/y)$ and $\theta_2 = \tan^{-1}(x_2/y)$.

28 •• A ring of radius R has a charge distribution on it that goes as $\lambda(\theta) = \lambda_0 \sin\theta$, as shown in the figure below. (*a*) In what direction does the field at the center of the ring point? (*b*) What is the magnitude of the field in the center of the ring?

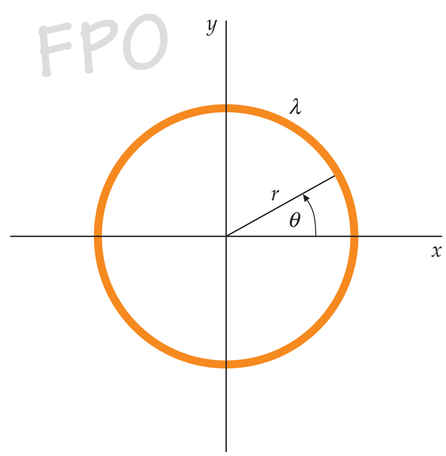


FIGURE 22-37
Problem 28

29 •• A finite line charge of uniform linear charge density λ lies on the x axis from $x = 0$ to $x = a$. Show that the y component of the electric field at a point on the y axis is given by

$$E_y = \frac{k\lambda}{y}\frac{a}{\sqrt{y^2 + a^2}}$$

30 ••• SSM A hemispherical thin shell of radius R carries a uniform surface charge σ. Find the electric field at the center of the hemispherical shell ($r = 0$).

Gauss's Law

31 • Consider a uniform electric field $\vec{E} = 2$ kN/C$\hat{\imath}$. (*a*) What is the flux of this field through a square of side 10 cm in a plane parallel to the yz plane? (*b*) What is the flux through the same square if the normal to its plane makes a 30° angle with the x axis?

32 • SSM A single point charge $q = +2\ \mu$C is at the origin. A spherical surface of radius 3.0 m has its center on the x axis at $x = 5$ m. (*a*) Sketch electric field lines for the point charge. Do any lines enter the spherical surface? (*b*) What is the net number of lines that cross the spherical surface, counting those that enter as negative? (*c*) What is the net flux of the electric field due to the point charge through the spherical surface?

33 • An electric field is $\vec{E} = 300\ \mathrm{N/C}\,\hat{i}$ for $x > 0$ and $\vec{E} = -300\ \mathrm{N/C}\,\hat{i}$ for $x < 0$. A cylinder of length 20 cm and radius 4 cm has its center at the origin and its axis along the x axis such that one end is at $x = +10$ cm and the other is at $x = -10$ cm. (*a*) What is the flux through each end? (*b*) What is the flux through the curved surface of the cylinder? (*c*) What is the net outward flux through the entire cylindrical surface? (*d*) What is the net charge inside the cylinder?

34 • Careful measurement of the electric field at the surface of a black box indicates that the net outward flux through the surface of the box is 6.0 kN·m²/C. (*a*) What is the net charge inside the box? (*b*) If the net outward flux through the surface of the box were zero, could you conclude that there were no charges inside the box? Why or why not?

35 • A point charge $q = +2\ \mu$C is at the center of a sphere of radius 0.5 m. (*a*) Find the surface area of the sphere. (*b*) Find the magnitude of the electric field at points on the surface of the sphere. (*c*) What is the flux of the electric field due to the point charge through the surface of the sphere? (*d*) Would your answer to Part (*c*) change if the point charge were moved so that it was inside the sphere but not at its center? (*e*) What is the net flux through a cube of side 1 m that encloses the sphere?

36 • SSM Since Newton's law of gravity and Coulomb's law have the same inverse-square dependence on distance, an expression analogous in form to Gauss's law can be found for gravity. The gravitational field $\vec{g}$ is the force per unit mass on a test mass m_0. Then, for a point mass m at the origin, the gravitational field g at some position r is

$$\vec{g} = -\frac{Gm}{r^2}\,\hat{r}$$

Compute the flux of the gravitational field through a spherical surface of radius R centered at the origin, and show that the gravitational analog of Gauss's law is $\phi_{\text{net}} = -4\pi G m_{\text{inside}}$

37 •• A charge of 2 μC is 20 cm above the center of a square of side length 40 cm. Find the flux through the square. (*Hint: Don't integrate.*)

38 •• In a particular region of the earth's atmosphere, the electric field above the earth's surface has been measured to be 150 N/C downward at an altitude of 250 m and 170 N/C downward at an altitude of 400 m. Calculate the volume charge density of the atmosphere assuming it to be uniform between 250 and 400 m. (You may neglect the curvature of the earth. Why?)

Spherical Symmetry

39 • A spherical shell of radius R_1 carries a total charge q_1 that is uniformly distributed on its surface. A second, larger spherical shell of radius R_2 that is concentric with the first carries a charge q_2 that is uniformly distributed on its surface. (*a*) Use Gauss's law to find the electric field in the regions $r < R_1$, $R_1 < r < R_2$, and $r > R_2$. (*b*) What should the ratio of the charges q_1/q_2 and their relative signs be for the electric field to be zero for $r > R_2$? (*c*) Sketch the electric field lines for the situation in Part (*b*) when q_1 is positive.

40 • A spherical shell of radius 6 cm carries a uniform surface charge density $\sigma = 9\ \mathrm{nC/m^2}$. (*a*) What is the total charge on the shell? Find the electric field at (*b*) $r = 2$ cm, (*c*) $r = 5.9$ cm, (*d*) $r = 6.1$ cm, and (*e*) $r = 10$ cm.

41 •• A sphere of radius 6 cm carries a uniform volume charge density $\rho = 450\ \mathrm{nC/m^3}$. (*a*) What is the total charge of the sphere? Find the electric field at (*b*) $r = 2$ cm, (*c*) $r = 5.9$ cm, (*d*) $r = 6.1$ cm, and (*e*) $r = 10$ cm. Compare your answers with Problem 40.

42 •• SSM Consider two concentric conducting spheres (Figure 22-38). The outer sphere is hollow and initially has a charge $-7Q$ deposited on it. The inner sphere is solid and has a charge $+2Q$ on it. (*a*) How is the charge distributed on the outer sphere? That is, how much charge is on the outer surface and how much charge is on the inner surface? (*b*) Suppose a wire is connected between the inner and outer spheres. After electrostatic equilibrium is established, how much total charge is on the outside sphere? How much charge is on the outer surface of the outside sphere, and how much charge is on the inner surface? Does the electric field at the surface of the inside sphere change when the wire is connected? If so, how? (*c*) Suppose we return to the original conditions in Part (*a*), with $+2Q$ on the inner sphere and $-7Q$ on the outer. We now connect the outer sphere to ground with a wire and then disconnect it. How much total charge will be on the outer sphere? How much charge will be on the inner surface of the outer sphere and how much will be on the outer surface?

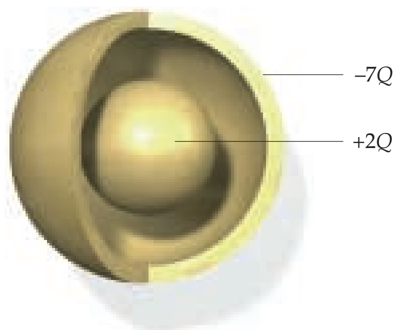


FIGURE 22-38
Problem 42

43 •• A nonconducting sphere of radius $R = 0.1$ m carries a uniform volume charge of charge density $\rho = 2.0\ \mathrm{nC/m^3}$. The magnitude of the electric field at $r = 2R$ is 1883 N/C. Find the magnitude of the electric field at $r = 0.5R$.

44 •• A nonconducting sphere of radius R carries a volume charge density that is proportional to the distance from the center: $\rho = Ar$ for $r \le R$, where A is a constant; $\rho = 0$ for $r > R$. (*a*) Find the total charge on the sphere by summing the charges on shells of thickness dr and volume $4\pi r^2\,dr$. (*b*) Find the electric field E_r both inside and outside the charge distribution, and sketch E_r versus r.

45 •• Repeat Problem 44 for a sphere with volume charge density $\rho = B/r$ for $r < R$; $\rho = 0$ for $r > R$.

46 •• SSM Repeat Problem 44 for a sphere with volume charge density $\rho = C/r^2$ for $r < R$; $\rho = 0$ for $r > R$.

47 ••• A thick, nonconducting spherical shell of inner radius a and outer radius b has a uniform volume charge density ρ. Find (*a*) the total charge and (*b*) the electric field everywhere.

Cylindrical Symmetry

48 •• Show that the electric field due to an infinitely long, uniformly charged cylindrical shell of radius R carrying a surface charge density σ is given by

$$E_r = 0, \qquad r < R$$

$$E_r = \frac{\sigma R}{\varepsilon_0 r} = \frac{\lambda}{2\pi\varepsilon_0 r} \qquad r > R$$

where $\lambda = 2\pi R\sigma$ is the charge per unit length on the shell.

49 •• A cylindrical shell of length 200 m and radius 6 cm carries a uniform surface charge density of $\sigma = 9\ \text{nC/m}^2$. (*a*) What is the total charge on the shell? Find the electric field at (*b*) $r = 2$ cm, (*c*) $r = 5.9$ cm, (*d*) $r = 6.1$ cm, and (*e*) $r = 10$ cm. (Use the results of Problem 48.)

50 •• An infinitely long nonconducting cylinder of radius R carries a uniform volume charge density of $\rho(r) = \rho_0$. Show that the electric field is given by

$$E_r = \frac{\rho R^2}{2\epsilon_0 r} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r} \qquad r > R$$

$$E_r = \frac{\rho}{2\epsilon_0}r = \frac{\lambda}{2\pi\epsilon_0 R^2}r \qquad r < R$$

where $\lambda = \rho\pi R^2$ is the charge per unit length.

51 •• A cylinder of length 200 m and radius 6 cm carries a uniform volume charge density of $\rho = 300\ \text{nC/m}^3$. (*a*) What is the total charge of the cylinder? Use the formulas given in Problem 50 to calculate the electric field at a point equidistant from the ends at (*b*) $r = 2$ cm, (*c*) $r = 5.9$ cm, (*d*) $r = 6.1$ cm, and (*e*) $r = 10$ cm. Compare your results with those in Problem 49.

52 •• SSM Consider two infinitely long, concentric cylindrical shells. The inner shell has a radius R_1 and carries a uniform surface charge density of σ_1, and the outer shell has a radius R_2 and carries a uniform surface charge density of σ_2. (*a*) Use Gauss's law to find the electric field in the regions $r < R_1$, $R_1 < r < R_2$, and $r > R_2$. (*b*) What is the ratio of the surface charge densities σ_2/σ_1 and their relative signs if the electric field is zero at $r > R_2$? What would the electric field between the shells be in this case? (*c*) Sketch the electric field lines for the situation in Part (*b*) if σ_1 is positive.

53 •• Figure 22-39 shows a portion of an infinitely long, concentric cable in cross section. The inner conductor carries a charge of 6 nC/m; the outer conductor is uncharged. (*a*) Find the electric field for all values of r, where r is the distance from the axis of the cylindrical system. (*b*) What are the surface charge densities on the inside and the outside surfaces of the outer conductor?

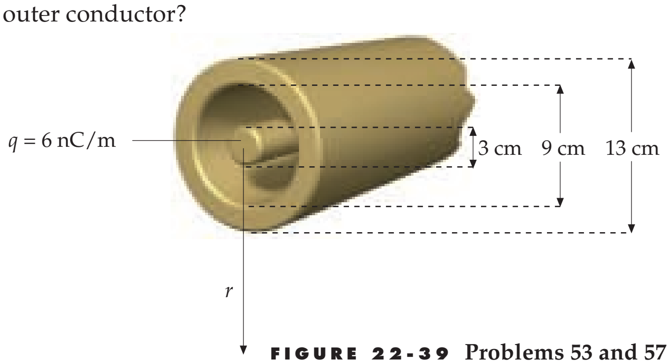


FIGURE 22-39 Problems 53 and 57

54 •• An infinitely long nonconducting cylinder of radius R and carrying a nonuniform volume charge density of $\rho(r) = ar$. (*a*) Show that the charge per unit length of the cylinder is $\lambda = 2\pi a R^3/3$. (*b*) Find the expressions for the electric field due to this charged cylinder. You should find one expression for the electric field in the region $r < R$ and a second expression for the field in the region $r > R$, as in Problem 50.

55 •• Repeat Problem 54 for a nonuniform volume charge density of $\rho = br^2$. In part (*a*) show $\lambda = \pi b R^4/2$ instead of the expression given for λ in Problem 54.

56 ••• An infinitely long, thick, nonconducting cylindrical shell of inner radius a and outer radius b has a uniform volume charge density ρ. Find the electric field everywhere.

57 ••• Suppose that the inner cylinder of Figure 22-39 is made of nonconducting material and carries a volume charge distribution given by $\rho(r) = C/r$, where $C = 200\ \text{nC/m}^2$. The outer cylinder is metallic. (*a*) Find the charge per meter carried by the inner cylinder. (*b*) Calculate the electric field for all values of r.

Charge and Field at Conductor Surfaces

58 • SSM A penny is in an external electric field of magnitude 1.6 kN/C directed perpendicular to its faces. (*a*) Find the charge density on each face of the penny, assuming the faces are planes. (*b*) If the radius of the penny is 1 cm, find the total charge on one face.

59 • An uncharged metal slab has square faces with 12-cm sides. It is placed in an external electric field that is perpendicular to its faces. The total charge induced on one of the faces is 1.2 nC. What is the magnitude of the electric field?

60 • A charge of 6 nC is placed uniformly on a square sheet of nonconducting material of side 20 cm in the yz plane. (*a*) What is the surface charge density σ? (*b*) What is the magnitude of the electric field just to the right and just to the left of the sheet? (*c*) The same charge is placed on a square conducting slab of side 20 cm and thickness 1 mm. What is the surface charge density σ? (Assume that the charge distributes itself uniformly on the large square surfaces.) (*d*) What is the magnitude of the electric field just to the right and just to the left of each face of the slab?

61 • A spherical conducting shell with zero net charge has an inner radius a and an outer radius b. A point charge q is placed at the center of the shell. (*a*) Use Gauss's law and the properties of conductors in equilibrium to find the electric field in the regions $r < a$, $a < r < b$, and $b < r$. (*b*) Draw the electric field lines for this situation. (*c*) Find the charge density on the inner surface ($r = a$) and on the outer surface ($r = b$) of the shell.

62 •• The electric field just above the surface of the earth has been measured to be 150 N/C downward. What total charge on the earth is implied by this measurement?

63 •• SSM A positive point charge of magnitude 2.5 μC is at the center of an uncharged spherical conducting shell of inner radius 60 cm and outer radius 90 cm. (*a*) Find the charge densities on the inner and outer surfaces of the shell and the total charge on each surface. (*b*) Find the electric field everywhere. (*c*) Repeat Part (*a*) and Part (*b*) with a net charge of +3.5 μC placed on the shell.

64 •• If the magnitude of an electric field in air is as great as 3×10^6 N/C, the air becomes ionized and begins to conduct electricity. This phenomenon is called dielectric breakdown. A charge of 18 μC is to be placed on a conducting sphere. What is the minimum radius of a sphere that can hold this charge without breakdown?

65 •• A square conducting slab with 5-m sides carries a net charge of 80 μC. (*a*) Find the charge density on each face of the slab and the electric field just outside one face of the slab. (*b*) The slab is placed to the right of an infinite charged nonconducting plane with charge density 2.0 μC/m^2 so that the faces of the slab are parallel to the plane. Find the electric field on each side of the slab far from its edges and the charge density on each face.

General Problems

66 •• Consider the three concentric metal spheres shown in Figure 22-40. Sphere one is solid, with radius R_1. Sphere two is hollow, with inner radius R_2 and outer radius R_3. Sphere three is hollow, with inner radius R_4 and outer radius R_5. Initially, all three spheres have zero excess charge. Then a negative charge $-Q_0$ is placed on sphere one and a positive charge $+Q_0$ is placed on sphere three. (*a*) After the charges have reached equilibrium, will the electric field in the space between spheres one and two point *toward* the center, *away* from the center, or neither? (*b*) How much charge will be on the inner surface of sphere two? Give the correct sign. (*c*) How much charge will be on the outer surface of sphere two? (*d*) How much charge will be on the inner surface of sphere three? (*e*) How much charge will be on the outer surface of sphere three? (*f*) Plot E versus r.

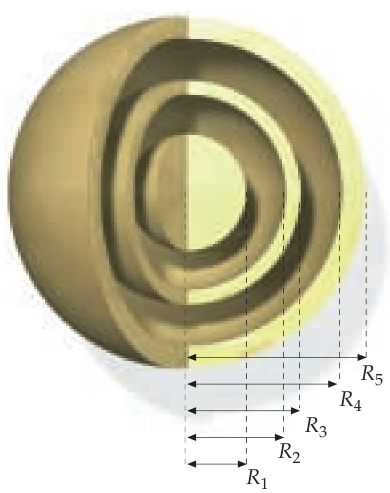


FIGURE 22-40 Problem 66

67 •• A nonuniform surface charge lies in the yz plane. At the origin, the surface charge density is $\sigma = 3.10$ μC/m^2. Other charged objects are present as well. Just to the right of the origin, the x component of the electric field is $E_x = 4.65 \times 10^5$ N/C. What is E_x just to the left of the origin?

68 •• An infinite line charge of uniform linear charge density $\lambda = -1.5$ μC/m lies parallel to the y axis at $x = -2$ m. A point charge of 1.3 μC is located at $x = 1$ m, $y = 2$ m. Find the electric field at $x = 2$ m, $y = 1.5$ m.

69 •• A thin nonconducting uniformly charged spherical shell of radius r (Figure 22-41*a*) has a total charge of Q. A small circular plug is removed from the surface. (*a*) What is the magnitude and direction of the electric field at the center of the hole? (*b*) The plug is put back in the hole (Figure 22-41*b*). Using the result of part *a*, calculate the force acting on the plug. (*c*) From this, calculate the "electrostatic pressure" (force/unit area) tending to expand the sphere.

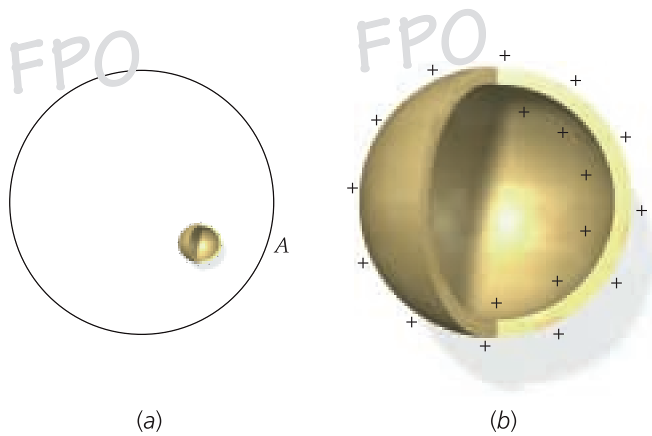


FIGURE 22-41 Problem 69

70 •• A soap bubble of radius $R_1 = 10$ cm has a charge of 3 nC uniformly spread over it. Because of electrostatic repulsion, the soap bubble expands until it bursts at a radius $R_2 = 20$ cm. From the results of Problem 69, calculate the work done by the electrostatic force in expanding the soap bubble.

71 •• If the soap bubble of Problem 70 collapses into a spherical water droplet, estimate the electric field at its surface.

72 •• Two infinite planes of charge lie parallel to each other and to the yz plane. One is at $x = -2$ m and has a surface charge density of $\sigma = -3.5$ μC/m^2. The other is at $x = 2$ m and has a surface charge density of $\sigma = 6.0$ μC/m^2. Find the electric field for (*a*) $x < -2$ m, (*b*) -2 m $< x < 2$ m, and (*c*) $x > 2$ m.

73 •• SSM An infinitely long cylindrical shell is coaxial with the y axis and has a radius of 15 cm. It carries a uniform surface charge density $\sigma = 6$ μC/m^2. A spherical shell of radius 25 cm is centered on the x axis at $x = 50$ cm and carries a uniform surface charge density $\sigma = -12$ μC/m^2. Calculate the magnitude and direction of the electric field at (*a*) the origin; (*b*) $x = 20$ cm, $y = 10$ cm; and (*c*) $x = 50$ cm, $y = 20$ cm. (See Problem 48.)

74 •• An infinite plane in the xz plane carries a uniform surface charge density $\sigma_1 = 65$ nC/m^2. A second infinite plane carrying a uniform charge density $\sigma_2 = 45$ nC/m^2 intersects the xz plane at the z axis and makes an angle of 30° with the xz plane, as shown in Figure 22-42. Find the electric field in the xy plane at (*a*) $x = 6$ m, $y = 2$ m and (*b*) $x = 6$ m, $y = 5$ m.

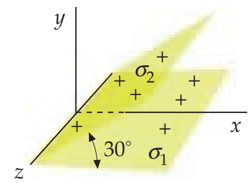


FIGURE 22-42 Problem 74

75 •• A quantum-mechanical treatment of the hydrogen atom shows that the electron in the atom can be treated as a smeared-out distribution of charge, which has the form: $\rho(r) = \rho_0 e^{-2r/a}$, where r is the distance from the nucleus, and a is the Bohr radius ($a = 0.0529$ nm). (*a*) Calculate ρ_0, from the fact that the atom is uncharged. (*b*) Calculate the electric field at any distance r from the nucleus. Treat the proton as a point charge.

76 •• SSM Using the results of Problem 75, if we placed a proton above the nucleus of a hydrogen atom, at what distance r would the electric force on the proton balance the gravitational force mg acting on it? From this result, explain why even though the electrostatic force is enormously stronger than the gravitational force, it is the gravitational force we notice more.

77 •• A ring of radius R carries a uniform, positive, linear charge density λ. Figure 22-43 shows a point P in the plane of the ring but not at the center. Consider the two elements of the ring of lengths s_1 and s_2 shown in the figure at distances r_1 and r_2, respectively, from point P. (*a*) What is the ratio of the charges of these elements? Which produces the greater field at point P? (*b*) What is the direction of the field at point P due to each element? What is the direction of the total electric field at point P? (*c*) Suppose that the electric field due to a point charge varied as $1/r$ rather than $1/r^2$. What would the electric field be at point P due to the elements shown? (*d*) How would your answers to Parts (*a*), (*b*), and (*c*) differ if point P were inside a spherical shell of uniform charge and the elements were of areas s_1 and s_2?

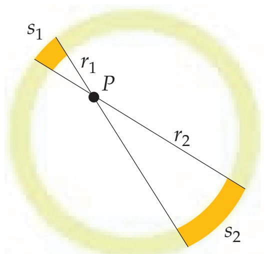


FIGURE 22-43 Problem 77

78 •• A uniformly charged ring of radius R that lies in a horizontal plane carries a charge Q. A particle of mass m carries a charge q, whose sign is opposite that of Q, is on the axis of the ring. (*a*) What is the minimum value of $|q|/m$ such that the particle will be in equilibrium under the action of gravity and the electrostatic force? (*b*) If $|q|/m$ is twice that calculated in Part (*a*), where will the particle be when it is in equilibrium?

79 •• A long, thin, nonconducting plastic rod is bent into a loop with radius R. Between the ends of the rod, a small gap of length l ($l << R$) remains. A charge Q is equally distributed on the rod. (*a*) Indicate the direction of the electric field at the center of the loop. (*b*) Find the magnitude of the electric field at the center of the loop.

80 •• A nonconducting sphere 1.2 m in diameter with its center on the x axis at $x = 4$ m carries a uniform volume charge of density $\rho = 5\ \mu C/m^3$. Surrounding the sphere is a spherical shell with a diameter of 2.4 m and a uniform surface charge density $\sigma = -1.5\ \mu C/m^2$. Calculate the magnitude and direction of the electric field at (*a*) $x = 4.5$ m, $y = 0$; (*b*) $x = 4.0$ m, $y = 1.1$ m; and (*c*) $x = 2.0$ m, $y = 3.0$ m.

81 •• An infinite plane of charge with surface charge density $\sigma_1 = 3\ \mu C/m^2$ is parallel to the xz plane at $y = -0.6$ m. A second infinite plane of charge with surface charge density $\sigma_2 = -2\ \mu C/m^2$ is parallel to the yz plane at $x = 1$ m. A sphere of radius 1 m with its center in the xy plane at the intersection of the two charged planes ($x = 1$ m, $y = -0.6$ m) has a surface charge density $\sigma_3 = -3\ \mu C/m^2$. Find the magnitude and direction of the electric field on the x axis at (*a*) $x = 0.4$ m and (*b*) $x = 2.5$ m.

82 •• An infinite plane lies parallel to the yz plane at $x = 2$ m and carries a uniform surface charge density $\sigma = 2\ \mu C/m^2$. An infinite line charge of uniform linear charge density $\lambda = 4\ \mu C/m$ passes through the origin at an angle of 45° with the x axis in the xy plane. A sphere of volume charge density $\rho = -6\ \mu C/m^3$ and radius 0.8 m is centered on the x axis at $x = 1$ m. Calculate the magnitude and direction of the electric field in the xy plane at $x = 1.5$ m, $y = 0.5$ m.

83 •• An infinite line charge λ is located along the z axis. A particle of mass m that carries a charge q whose sign is opposite to that of λ is in a circular orbit in the xy plane about the line charge. Obtain an expression for the period of the orbit in terms of m, q, R, and λ, where R is the radius of the orbit.

84 •• SSM A ring of radius R that lies in the yz plane carries a positive charge Q uniformly distributed over its length. A particle of mass m that carries a negative charge of magnitude q is at the center of the ring. (*a*) Show that if $x << R$, the electric field along the axis of the ring is proportional to x. (*b*) Find the force on the particle of mass m as a function of x. (*c*) Show that if m is given a small displacement in the x direction, it will perform simple harmonic motion. Calculate the period of that motion.

85 •• When the charges Q and q of Problem 84 are 5 μC and $-5\ \mu$C, respectively, and the radius of the ring is 8.0 cm, the mass m oscillates about its equilibrium position with an angular frequency of 21 rad/s. Find the angular frequency of oscillation of the mass if the radius of the ring is doubled to 16 cm and all other parameters remain unchanged.

86 •• Given the initial conditions of Problem 85, find the angular frequency of oscillation of the mass if the radius of the ring is doubled to 16 cm while keeping the linear charge density on the ring constant.

87 •• A uniformly charged nonconducting sphere of radius a with center at the origin has volume charge density ρ. (a) Show that at a point within the sphere a distance r from the center $\vec{E} = \frac{\rho}{3\epsilon_0} r\,\hat{r}$. ($b$) Material is removed from the sphere leaving a spherical cavity of radius $b = a/2$ with its center at $x = b$ on the x axis (Figure 22-44). Calculate the electric field at points 1 and 2 shown in Figure 22-44. (*Hint: Replace the sphere-with-cavity with two uniform spheres of equal positive and negative charge densities.*)

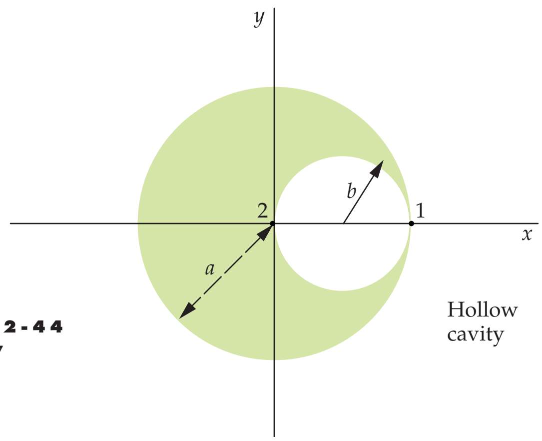


FIGURE 22-44
Problem 87

88 ••• Show that the electric field throughout the cavity of Problem 87 is uniform and is given by

$$\vec{E} = \frac{\rho}{3\epsilon_0} b\,\hat{i}$$

89 •• Repeat Problem 87 assuming that the cavity is filled with a uniformly charged material wth a total charge of Q.

90 •• A nonconducting cylinder of radius 1.2 m and length 2.0 m carries a charge of 50 μC uniformly distributed throughout the cylinder. Find the electric field *on the cylinder axis* at a distance of (a) 0.5 m, (b) 2.0 m, and (c) 20 m from the center of the cylinder.

91 •• A uniform line charge of density λ lies on the x axis between $x = 0$ and $x = L$. Its total charge is $Q = 8$ nC. The electric field at $x = 2L$ is 600 N/C$\,\hat{i}$. Find the electric field at $x = 3L$.

92 ••• A *small* gaussian surface in the shape of a cube with faces parallel to the xy, xz, and yz planes (Figure 23-45) is in a region in which the electric field remains parallel with the x axis. Using the Taylor series (and neglecting terms higher than first order), show that the net flux of the electric field out of the gaussian surface is given by

$$\phi_{\text{net}} = \frac{\partial E_x}{\partial x}\Delta V$$

where ΔV is the volume enclosed by the gaussian surface.

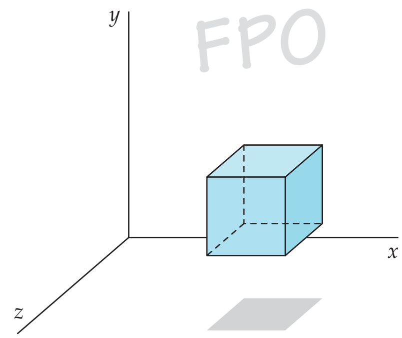


FIGURE 22-45 Problem 92

Remark: The corresponding result for situations for which the direction of the electric field is not restricted to one dimension is

$$\phi_{\text{net}} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\Delta V$$

where the combination of derivatives in the parentheses is commonly written $\vec{\nabla}\cdot\vec{E}$ and is called the *divergence* of $\vec{E}$.

93 •• Using Gauss's law and the results of Problem 92 show that

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

where ρ is the volume charge density. (This equation is known as the point form of Gauss's law.)

94 ••• SSM A dipole $\vec{p}$ is located at a distance r from an infinitely long line charge with a uniform linear charge density λ. Assume that the dipole is aligned with the field due to the line charge. Determine the force that acts on the dipole.

95 •• Consider a simple but surprisingly accurate model for the Hydrogen molecule: two positive point charges, each with charge $+e$, are placed inside a sphere of radius R, which has uniform charge density $-2e$. The two point charges are placed symmetrically (Figure 22-46). Find the distance from the center, a, where the net force on either charge is 0.

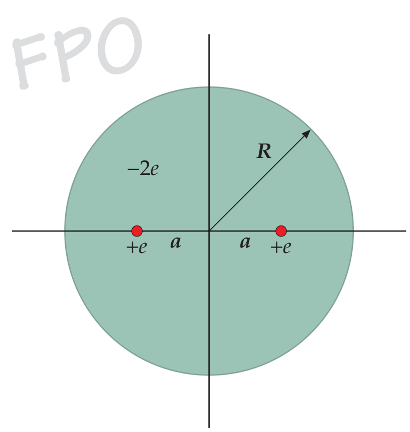


FIGURE 22-46
Problem 95