Institutt for fysikk, NTNU
TFY4155/FY1003: Elektrisitet og magnetisme
Spring 2005

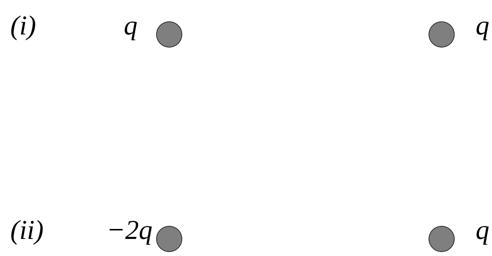
Øving 3

Guidance: January 27. and 28.
Deliver no later than: Monday January 31

Exercise 1

a) In Øving 2, we determined the electric field from a uniformly charged rod and a uniformly charged circular disk. For the rod and the disk in Øving 2, try to sketch electric field lines, both in a plane that contains the rod (disk) and in a plane perpendicular to the rod (disk) through its center. Draw sketches both in a large and a small scale in each of the four separate cases, so that they give a qualitative picture of the field, both close to and far away from the rod (disk).

b) Sketch electric field lines for these two systems of point charges:



Hint: In this exercise, it may be of some help to have a look at http://www.falstad.com/vector3de, a Java applet for visualization of (among other things) electric field lines ("Display: Field Lines") from various charge distributions (both point charges and continuous charge distributions). "finite line" represents exactly the charged rod. However, the closest we get to the circular disk is "charged plate", which has a finite extent in the x direction. The length of the rod and the size of the disk may be changed with the bottom scrollbar.

Exercise 2

In cartesian coordinates, a differential line element is represented by the vector

$$d\boldsymbol{l} = dx\ \hat{x} + dy\ \hat{y} + dz\ \hat{z}$$

A differential volume element is (the scalar)

$$dV = dx\ dy\ dz$$

A differential surface element may be expressed in terms of a *vector*: Just as a line element must be represented by an *absolute value* (i.e., the length of the element) and a *direction*, a surface element must be specified in terms of an absolute value (i.e., the area of the surface) and an *orientation*. And in order to specify the orientation of the surface element, it appears rather natural to choose the direction in space that is perpendicular to the surface; the socalled surface normal. This implies that a surface element of area dA being oriented in space such the unit normal $\hat{n}$ (dimensionless, and with length 1) is perpendicular to the surface, may be represented by the vector

$$dA\ \hat{n}$$

In other words: A vector with direction normal to the surface, with absolute value equal to the surface area dA, and with the dimension of a length squared. (I.e., with the unit m^2.)
For simplicity, we often use the notation

$$d\boldsymbol{A} = dA\ \hat{n}$$

for such a surface element.
In cartesian coordinates, then, we have the following three differential surfaces, with orientations such that their surface normals point in the direction of the x-, y- and z-axis, respectively:

$$\begin{aligned} d\boldsymbol{A}_x &= dy\ dz\ \hat{x} \\ d\boldsymbol{A}_y &= dx\ dz\ \hat{y} \\ d\boldsymbol{A}_z &= dx\ dy\ \hat{z} \end{aligned}$$

a) Show that in spherical coordinates (r, θ, ϕ), the corresponding quantities are

$$\begin{aligned} d\boldsymbol{l} &= dr\ \hat{r} + r\ d\theta\ \hat{\theta} + r \sin\theta\ d\phi\ \hat{\phi} \\ d\boldsymbol{A}_r &= r^2 \sin\theta\ d\theta\ d\phi\ \hat{r} \\ d\boldsymbol{A}_\theta &= r\ dr \sin\theta\ d\phi\ \hat{\theta} \\ d\boldsymbol{A}_\phi &= r\ dr\ d\theta\ \hat{\phi} \\ dV &= r^2\ dr \sin\theta\ d\theta\ d\phi \end{aligned}$$

In other words: The surface element $d\boldsymbol{A}_r$ is oriented such that its surface normal points radially outwards, i.e., along $\hat{r}$, and correspondingly for the other two.

Hint: You will find it helpful to draw a differential volume element, limited by r and $r + dr$, θ and $\theta + d\theta$, ϕ and $\phi + d\phi$. See the figure in "ukentlig sammendrag", week 3. Note that the three orthogonal unit vectors $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are *not* fixed vectors in space (as opposed to the cartesian unit vectors $\hat{x}$, $\hat{y}$ and $\hat{z}$), they depend on *where* in space we are, and they *change direction* as we move around. Examples: In a point on the y-axis (with $y > 0$), $\hat{r} = \hat{y}$, while in a point on the z-axis ($z > 0$), $\hat{r} = \hat{z}$. In the whole xy plane, we have $\hat{\theta} = -\hat{z}$, while $\hat{\phi} = -\hat{x}$ on the positive y axis and $\hat{\phi} = \hat{x}$ on the negative y axis.

b) Based on the formulas given above, show that a sphere with radius R has a volume $4\pi R^3/3$ and a surface area $4\pi R^2$.

Comment: The circumference of a circle of radius R is $2\pi R$. The circle *spans an angle* 2π (radians). We arrive at this result, starting with a differential element $dl = R\, d\theta$ and integrating from 0 to 2π. Analogously, a sphere of radius R has a surface area $4\pi R^2$. The sphere *spans a solid angle* 4π (socalled steradians). Note the analogy between angle in the plane and solid angle in 3D space! In the latter case, the starting point is a differential *surface element* $dA_R = R^2\, d\Omega$, where $d\Omega = \sin\theta\, d\theta\, d\phi$ is a differential *solid angle element*. Integration over θ (from "north", i.e., along the positive z-axis, i.e., $\theta = 0$, to "south", i.e., along the negative z-axis, i.e., $\theta = \pi$) and ϕ (from a direction along the positive x-axis, i.e., $\phi = 0$, and one complete turn, i.e., to $\phi = 2\pi$) then yields the total surface area A of the sphere, or the total solid angle $\Omega = 4\pi$. We will soon come back to this in the lectures, in connection to *Gauss' law*.

c) Suppose now that we have a sphere of radius R. Inside the sphere, we have electric charge given by the charge density (i.e., charge pr unit volume)

$$\rho(r, \theta) = \rho_0 \frac{r}{R} \cos^2\theta$$

Where do we have the highest value of the charge density? And the smallest? Determine the total charge Q of the sphere.