**Institutt for fysikk, NTNU**
**TFY4155/FY1003 Elektrisitet og magnetisme**
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# Solution to øving 4

Guidance February 3 and 4

*Exercise 1*

The potential difference $\Delta V$ between two points in space is given by

$$\Delta V = V_B - V_A = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l}$$

In this exercise, we have a uniform electric field $\boldsymbol{E} = E_0\,\hat{x}$, so we may write

$$\Delta V = -E_0\,\hat{x} \cdot \int_A^B d\boldsymbol{l}$$

where $A$ represents the origin, $(0,0)$, and $B$ the three points given in the text. We find:
$(i)$

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(a,0)} d\boldsymbol{l} = a\,\hat{x}$$

hence

$$\Delta V = -E_0\,\hat{x} \cdot a\,\hat{x} = -E_0 a$$

$(ii)$

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(0,a)} d\boldsymbol{l} = a\,\hat{y}$$

hence

$$\Delta V = -E_0\,\hat{x} \cdot a\,\hat{y} = 0$$

$(iii)$

$$\int_A^B d\boldsymbol{l} = \int_{(0,0)}^{(a,a)} d\boldsymbol{l} = a\,\hat{x} + a\,\hat{y}$$

hence

$$\Delta V = -E_0\,\hat{x} \cdot (a\,\hat{x} + a\,\hat{y}) = -E_0 a$$

*Exercise 2*

$a)$ With our choice for the polar angle $\theta$, we see from the figure that

$$\begin{aligned} x &= r\sin\theta \\ z &= r\cos\theta \\ r &= \sqrt{x^2+z^2} \end{aligned}$$

$b$) We use the superposition principle to determine the potential from the two point charges. With the point $(x, z)$ in a distance $r_1$ from $q$ and $r_2$ from $-q$, we have

$$\begin{aligned} V(x,z) &= \frac{q}{4\pi\varepsilon_0 r_1} - \frac{q}{4\pi\varepsilon_0 r_2} \\ &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+(z-a/2)^2}} - \frac{1}{\sqrt{x^2+(z+a/2)^2}}\right) \end{aligned}$$

The distances $r_1$ and $r_2$ in terms of $x$ and $z$ are found directly by looking at the figure.

The potential on the $x$ axis is

$$V(x,0) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{x^2+a^2/4}} - \frac{1}{\sqrt{x^2+a^2/4}}\right) = 0$$

The potential on the $z$ axis is

$$V(0,z) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|}\right)$$

Note that we must take absolute values in order to have *one* expression valid *throughout* the $z$ axis. With $z > a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = \frac{1}{z-a/2} - \frac{1}{z+a/2} = \frac{a}{z^2-a^2/4}$$

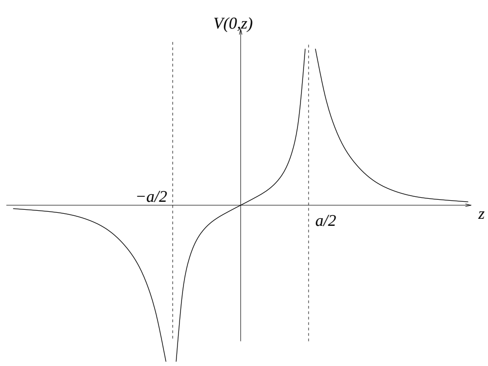
With $z < -a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = -\frac{1}{z-a/2} + \frac{1}{z+a/2} = -\frac{a}{z^2-a^2/4}$$

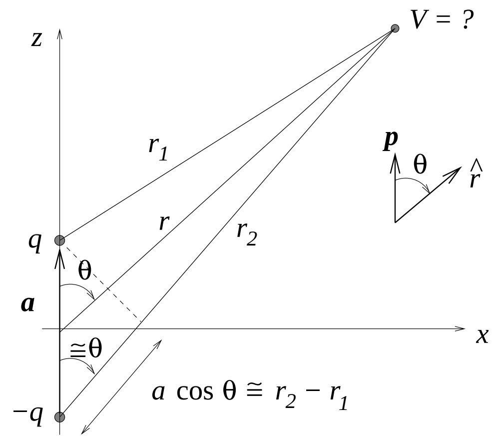
With $-a/2 < z < a/2$:

$$\frac{1}{|z-a/2|} - \frac{1}{|z+a/2|} = -\frac{1}{z-a/2} - \frac{1}{z+a/2} = -\frac{2z}{z^2-a^2/4} = \frac{2z}{a^2/4-z^2}$$

A sketch of $V(0,z)$:



$c$) We use the hint given in the text, in addition to the following figure, and obtain:




$$
\begin{aligned}
V(r,\theta) &= \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) \\
&= \frac{q}{4\pi\varepsilon_0}\cdot\frac{r_2-r_1}{r_1 r_2} \\
&\simeq \frac{q}{4\pi\varepsilon_0}\cdot\frac{a\cos\theta}{r^2} \\
&= \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} \\
&= \frac{pr\cos\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\varepsilon_0 r^3}
\end{aligned}
$$

Alternatively, proceeding a bit more slowly: From the figure, we see that

$$
\begin{aligned}
r_1 &\simeq r-\frac{a}{2}\cos\theta \\
r_2 &\simeq r+\frac{a}{2}\cos\theta
\end{aligned}
$$

When $r \gg a$ we may expand both $1/r_1$ and $1/r_2$ in power series around $1/r$:

$$
\begin{aligned}
\frac{1}{r_1}-\frac{1}{r_2} &\simeq \frac{1}{r-\frac{a}{2}\cos\theta}-\frac{1}{r+\frac{a}{2}\cos\theta} \\
&= \frac{1}{r}\left[\left(1-\frac{a\cos\theta}{2r}\right)^{-1}-\left(1+\frac{a\cos\theta}{2r}\right)^{-1}\right] \\
&\simeq \frac{1}{r}\left[1+\frac{a\cos\theta}{2r}-1+\frac{a\cos\theta}{2r}\right] \\
&= \frac{a\cos\theta}{r^2}
\end{aligned}
$$

Is it *reasonable* that the potential from an electric dipole falls off faster than the potential from a point charge (i.e., an electric "monopole")? Yes, because the negative and the positive point charges of the dipole contribute with opposite signs to the total potential. Thus, the contributions to the potential from the two point charges partly cancel each other. (On the $x$ axis, the two contributions cancel *exactly*.)

Extra, if you wonder how one should proceed in order to find the dominating *correction* to the result obtained above:
A first thought might be to continue the series expansion that was started above, and include sufficiently many terms so that a dominating correction was obtained. If we include one more term, nothing new is obtained since that term comes with the same sign in the expansion of the two square roots, and therefore (with the minus sign in front of one of them) cancel each other. We must include *two* more terms:

$$
\begin{aligned}
& \frac{1}{r}\left[\left(1-\frac{a\cos\theta}{2r}\right)^{-1}-\left(1+\frac{a\cos\theta}{2r}\right)^{-1}\right] \\
= & \frac{1}{r}\left[1+\frac{a\cos\theta}{2r}+\left(\frac{a\cos\theta}{2r}\right)^2+\left(\frac{a\cos\theta}{2r}\right)^3+\ldots-\left(1-\frac{a\cos\theta}{2r}+\left(\frac{a\cos\theta}{2r}\right)^2-\left(\frac{a\cos\theta}{2r}\right)^3+\ldots\right)\right] \\
= & \frac{a\cos\theta}{r^2}-\frac{a^3\cos^3\theta}{4r^4}+\ldots \\
= & \frac{a\cos\theta}{r^2}\left(1-\frac{a^2\cos^2\theta}{4r^2}+\ldots\right)
\end{aligned}
$$

Here, we used the series expansion $(1+x)^{-1}=1-x+x^2-x^3+\ldots$ (valid for $|x|<1$). Not a bad try. However, there is a catch here: The *starting point* for this series expansion was an approximation itself, namely

$$
\begin{aligned}
r_1 &\simeq r-\frac{a}{2}\cos\theta \\
r_2 &\simeq r+\frac{a}{2}\cos\theta
\end{aligned}
$$

And the errors we do in these approximations are of the same order of magnitude as the correction term we are looking for!
The solution is obtained by going back to the exact expression for $V$, with $r_1$ and $r_2$ expressed in terms of cartesian coordinates $x$ and $z$. The calculation is not difficult, but rather tedious, so the details are skipped here. If my calculation is correct, the answer is

$$
\frac{1}{r_1}-\frac{1}{r_2}=\frac{a\cos\theta}{r^2}\left[1-\frac{3a^2}{8r^2}\left(1-\frac{5}{3}\cos^2\theta\right)+\ldots\right]
$$

Here, we have included all corrections that are an order $a^2/r^2$ smaller than the dominating result. The next term in this series will be further reduced, by an additional factor $a^2/r^2$. In series expansions like this, the first term that is *not* included will always be smaller than the last term that we did include, since the series is a polynomial with increasing powers of a parameter which is small compared to 1. (In our particular case, we see that for directions given by $\cos^2\theta\simeq 3/5$, the first correction actually disappears. In that case, one may have to look at the next term in the series expansion.)

*Exercise 3*

*a)* **C**

$$\boldsymbol{F} = q\boldsymbol{E} = m\boldsymbol{a}$$

Newton's 2. law! Here, $q = -e$, so the acceleration of the electron becomes

$$\boldsymbol{a} = -\frac{e}{m}\boldsymbol{E}$$

i.e., to the left.

*b)* **C**

$$\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

provided that

$$d\boldsymbol{l} \perp \boldsymbol{E}$$

*c)* **B**
You know that a body initially at rest will fall in the gravitational field of the earth. In other words, it moves in the direction of lower potential energy. In the same way, a charged particle initially at rest will move in the direction of lower potential energy in an electric field. Mathematically: ($\boldsymbol{F}$ = force, $q$ = charge ($q < 0$), $U$ = potential energy, $V$ = potential)

$$\begin{aligned}
\boldsymbol{F} &= q\boldsymbol{E} = -|q|\boldsymbol{E} \\
\boldsymbol{E} &= -\nabla V \\
U &= qV = -|q|V \\
\boldsymbol{F} &= -\nabla U = -q\nabla V = |q|\nabla V
\end{aligned}$$

The movement of the particle must obviously be in the same direction as $\boldsymbol{F}$ (given that the initial velocity is zero), so we see that the movement will be in the opposite direction of $\boldsymbol{E}$, and in the direction of higher potential $V$.

*d)* **D**
The total potential energy of a system of point charges is

$$U = \sum_{i<j} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$$

where the sum is taken over all pairs of charges $q_i$ and $q_j$ separated by distances $r_{ij}$. In our case, all charges have the same absolute value. We have 4 pairs with opposite sign, where the charges are 5 cm away from each other, and 2 pairs (diagonally) with equal sign, where the charges are $\sqrt{50}$ cm away from each other. Hence, we obtain

$$U = 9 \cdot 10^9 \cdot (9 \cdot 10^{-6})^2 \cdot \left[-\frac{4}{0.05} + \frac{2}{\sqrt{50} \cdot 10^{-2}}\right] \simeq -38 \text{ J}$$

*e)* **D**

The point charges $Q_1$ and $Q_2$ are not being moved, so we do not have to consider the mutual potential energy for this pair when only the third charge (the electron) is moved. We must calculate the potential energy due to the interaction between the electron and the two fixed charges, before and after the displacement, respectively. Alternatively, we may start by calculating the potential from the charges $Q_1$ and $Q_2$ in the positions A and B, i.e., $V_A$ and $V_B$, and next find the change in potential energy, $\Delta U = U_B - U_A = -eV_B - (-e)V_A = -e(V_B - V_A)$. The potential in distance $r$ from a point charge $q$ is

$$V(r) = \frac{q}{4\pi\varepsilon_0 r}$$

i.e., the Coulomb potential. The relevant separations in this case are 0.6 m (from $Q_1$ to A and from $Q_2$ to B) and $\sqrt{0.6^2 + 0.8^2} = 1.0$ m (from $Q_1$ to B and from $Q_2$ to A). Thus,

$$V_A = \frac{Q_1}{4\pi\varepsilon_0 \cdot 0.6} + \frac{Q_2}{4\pi\varepsilon_0 \cdot 1.0}$$

and

$$V_B = \frac{Q_1}{4\pi\varepsilon_0 \cdot 1.0} + \frac{Q_2}{4\pi\varepsilon_0 \cdot 0.6}$$

which yields

$$\Delta V = V_B - V_A = -\frac{2(Q_1 - Q_2)}{3 \cdot 4\pi\varepsilon_0} = -\frac{2}{3} \cdot 9 \cdot 10^9 \cdot (69 + 98) \cdot 10^{-9} = -1002 \text{ V}$$

and finally

$$\Delta U = -e \cdot \Delta V \simeq +1 \text{ keV}$$

*f)* **B**

$$U = \frac{e^2}{4\pi\varepsilon_0 r} = e \cdot \frac{e}{4\pi\varepsilon_0 r} = e \cdot 1.6 \cdot 10^{-19} \cdot 9 \cdot 10^9 \cdot 10^{10} = 14.4 \text{ eV}$$

*g)* **D**

Energy conservation yields

$$\frac{1}{2}mv^2 = qV$$

I.e., acceleration of a particle with charge $q$ and mass $m$ through a potensial difference $V$ results in a reduction of potential energy, $qV$, and a corresponding increase in kinetic energy, $mv^2/2$. Equal speed for the two particles then implies

$$\frac{q_\alpha V_\alpha}{m_\alpha} = \frac{q_{\text{Be}} V_{\text{Be}}}{m_{\text{Be}}}$$

in other words

$$\frac{V_{\text{Be}}}{V_\alpha} = \frac{q_\alpha m_{\text{Be}}}{q_{\text{Be}} m_\alpha} = \frac{2 \cdot 9}{4 \cdot 4} = \frac{9}{8}$$