Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Vår 2005

Solution to øving 8

Guidance week 9 and 10.

Question	A	B	C	D
1				x
2			x	
3			x	
4		x		
5	x			
6	x			
7	x			
8	x			
9				x
10			x	
11	x			
12		x		
13		x		
14	x			
15			x	
16				x
17				x
18			x	
19		x		
20	x			

Comment: In this øving, you were given several hints. On the midterm exam, you may not be given so many hints like these.

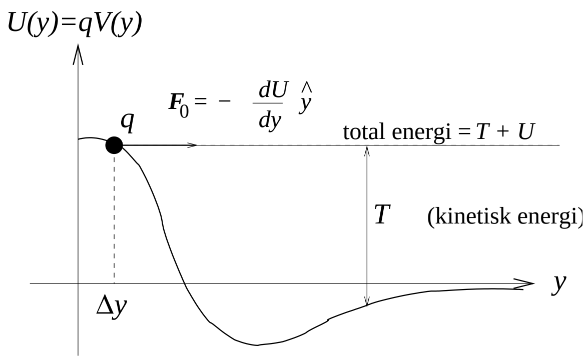
1) Obviously, B is a possible solution: If $b \to 0$, we have essentially two point charges $-Q$ in $x = \pm a$, and the charge q is pulled towards the origin. The energy of the point charge will change between potential and kinetic, with the lowest potential energy and the largest kinetic energy in the origin, and zero kinetic energy in $y = \pm \Delta y$.
Next, consider if A is possible: First thought might suggest no, because when q comes very far away, it sees essentially a point charge $-2Q + Q + Q - 2Q = -2Q$, and thereby feels an attractive force, i.e., directed back towards the origin. This way of thinking would have been correct if q started with zero velocity far out on the y axis. However, q starts with zero velocity

in $y = \Delta y$. Now, if we let b become very large, the potential from the charges on the x axis is approximately

$$V(\Delta y) \simeq \frac{2Q}{4\pi\varepsilon_0 a}$$

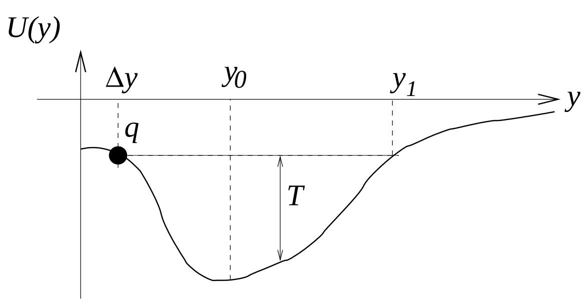
Here, we have neglected the contribution from the two negative charges, since $b \gg a$. We have also assumed $\Delta y \ll a$ and approximated the distance between q and Q with a. (The latter was not really necessary.)

Here comes the point: We have $V(\Delta y) > 0$, and hence a *positive* potential energy for the charge q: $U = qV(\Delta y)$. With zero initial velocity, this is also the total energy of the charge. In the limit $y \to \infty$, we have zero potential, hence zero potential energy for the charge q. Because of energy conservation, this means that q has a *positive kinetic energy* for all $y > \Delta y$, and therefore it will disappear towards $y = \infty$. The following figure illustrates the energy relations for the charge q when $b \gg a$:



If you are convinced that both A and B are possible outcomes, you don't really need to consider C: The correct answer must be D.

However, from the figure we may also realize that C is a possible outcome: As we reduce the value of b, we must reach a point where $V(\Delta y) = 0$. Then the total energy is exactly big enough for the charge q to disappear to infinity. A further reduction in b results in a negative value of $V(\Delta y)$, and hence a negative total energy. In that case, the total energy is simply not big enough for q to "escape". It will start moving out on the y axis until it reaches a position y_1 where V equals $V(\Delta y)$. In this position, we must have zero kinetic energy, $T = 0$. The charge reverses its direction and returns to its starting point, and so it continues to oscillate back and forth.



If we have friction, the particle will lose some energy (as heat to its surroundings) and perform oscillations with smaller and smaller "amplitude". In the end, it will stop in the equilibrium position y_0.

2) The electron mass m_e and the proton mass m_p are given, so it's just a question of evaluating the ratio. This is in fact quite important, and something one encounters many times in other physics courses: Since the mass of the electron is so small, they will move much faster than the atomic nuclei. From an electron's point of view, the nuclei essentially do not move! Many properties of atoms, molecules, and solids can, with excellent approximation, be calculated with such a simplifying starting point.

3) We know that a charged metal ball would choose alternative 1, namely net charge uniformly distributed over its surface, and not over its volume. We have proven this via Gauss' law. However, it must also be so that this corresponds to the lowest possible potential energy. Hence, we must have $U_1 < U_2$, which is only true for alternative C.
How to calculate U_2? In the lectures, we calculated U_1 in two different ways: First by considering the work necessary to charge the sphere from zero to final charge Q. The work required to increase the charge from q to $q + dq$ is $dW = v(q)dq$, where $v(q)$ is the potential on the sphere when it has charge q. Hence,

$$U = \int_0^Q v(q)\, dq$$

Alternatively, we could use that the energy pr unit volume in an electric field is $u = \varepsilon_0 E^2/2$, so that

$$U = \int_{\text{all space}} u\, dV = \int_{\text{all space}} \frac{1}{2}\varepsilon_0 E^2\, dV$$

(See question 11 in øving 9, where this is done for U_1.)
Let us find U_2 with both methods! First, we imagine that we "create" the uniformly charged sphere by starting at the center, and then adding small spherical shells. At some stage in this process, we have uniform charge density inside radius r, i.e., total charge $q = Qr^3/R^3$. The electric field outside r is now $E(r') = q/4\pi\varepsilon_0 r'^2$, so the potential v at r is

$$v(r) = -\int_\infty^r E(r')\, dr' = \frac{q}{4\pi\varepsilon_0 r} = \frac{Qr^3/R^3}{4\pi\varepsilon_0 r}$$

The work required to add a new thin shell of thickness dr and charge dq is therefore

$$dW = v(r)\, dq = v(r)\rho\, dV = v(r)\frac{Q}{4\pi R^3/3}\, 4\pi r^2\, dr$$

since the charge density ρ is uniform, i.e., equal to the total charge Q divided by the total volume $4\pi R^3/3$. The total work required to create the whole sphere must be the integral of dW, and this is also equal to the total potential energy U_2:

$$\begin{aligned} U_2 &= \int dW \\ &= \int v(r)\, dq \\ &= \int v(r)\frac{Q}{4\pi R^3/3}\, 4\pi r^2\, dr \end{aligned}$$

$$
\begin{aligned}
&= \int_0^R \frac{Qr^3/R^3}{4\pi\varepsilon_0 r}\frac{Q}{4\pi R^3/3}\,4\pi r^2\,dr \\
&= \frac{3Q^2}{4\pi\varepsilon_0 R^6}\int_0^R r^4\,dr \\
&= \frac{3Q^2}{20\pi\varepsilon_0 R}
\end{aligned}
$$

Alternatively, by integrating the energy density expressed in terms of the electric field: Using Gauss' law, we find that the electric field inside the sphere (i.e. $r < R$) is $E(r) = Qr/4\pi\varepsilon_0 R^3$ and outside the sphere (i.e. $r > R$) it is $E(r) = Q/4\pi\varepsilon_0 r^2$. Hence, the potential energy U_2 is:

$$
\begin{aligned}
U_2 &= \int_{\text{all space}} \frac{1}{2}\varepsilon_0 E^2\,dV \\
&= \int_0^R \frac{1}{2}\varepsilon_0\left(\frac{Qr}{4\pi\varepsilon_0 R^3}\right)^2 4\pi r^2\,dr + \int_R^\infty \frac{1}{2}\varepsilon_0\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)^2 4\pi r^2\,dr \\
&= \frac{1}{2}\varepsilon_0\frac{Q^2}{4\pi\varepsilon_0^2 R^6}\cdot\frac{R^5}{5} + \frac{1}{2}\varepsilon_0\frac{Q^2}{4\pi\varepsilon_0^2}\cdot\frac{1}{R} \\
&= \frac{Q^2}{8\pi\varepsilon_0 R}\left(\frac{1}{5}+1\right) \\
&= \frac{3Q^2}{20\pi\varepsilon_0 R}
\end{aligned}
$$

The same answer!

4) A charged metal sphere will attract a sphere with the opposite charge. However, it will also attract a neutral metal sphere because of polarization of the neutral sphere. (See further discussion in question 9 below.) So, it is *always* true that at least one of the spheres is charged. (While it *may* be true that both spheres are charged.)

5) Gauss' law yields

$$
\begin{aligned}
E(r)\cdot 4\pi r^2 &= \frac{1}{\varepsilon_0}\int_0^r \rho(r')\,dV' \\
&= \frac{k}{\varepsilon_0}\int_0^r \frac{1}{r'}\cdot 4\pi (r')^2\,dr' \\
&= \frac{4\pi k}{\varepsilon_0}\bigg|_0^r \frac{1}{2}(r')^2 \\
&= \frac{2\pi k}{\varepsilon_0} r^2
\end{aligned}
$$

for $r < R$. Hence

$$E(r) = \frac{k}{2\varepsilon_0}$$

i.e., constant field strength for $r < R$. For $r > R$, the field becomes the same as for a point charge in the origin, i.e., proportional to $1/r^2$, the same in all four graphs. Please, note the "simplifications" that may be done when we have spherical symmetry, and therefore choose a spherical Gaussian surface: The surface integral of the electric field simply becomes $E(r) \cdot 4\pi r^2$, whereas the volume element becomes $dV = 4\pi r^2 \, dr$. In both cases, the integration over the directions, i.e., the angles θ and ϕ, only gave a factor of 4π.

6) The glass rod does not touch the metal spheres, so there can be no transfer of charge between the glass rod and the spheres. However, the positively charged glass rod will attract free electrons in the metal, resulting in an excess of electrons on the left side of the left metal sphere. Since the metal spheres are overall electrically neutral, this implies that the right sphere must end up with a deficiency of electrons, i.e., a net positive charge. This is again polarization (see questions 4, 9, and 14), or "charging by induction" (see textbooks).

7) Since the capacitance C pr definition is given by the ratio between the charge Q on the capacitor and the potential difference ΔV between the two conductors that make up the capacitor, we must calculate ΔV between the two spherical shells. The electric field between the two spherical shells is

$$\boldsymbol{E}(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$$

which is found by Gauss' law. (With some experience, one should be able to *remember* that electric field from a spherically symmetric charge distribution is determined by the charge "inside" of where we are, and behaves as if all this charge was located in the centre - just like the gravitational field from a spherically symmetric mass distribution, e.g. the earth.)
Then, we may use

$$\Delta V = V(a) - V(b) = -\int_b^a E(r)\, dr$$

to find the potential difference ΔV. (We integrate along a radially directed curve, so that $\boldsymbol{E} \cdot d\boldsymbol{l} = E(r)\, dr(\hat{r} \cdot \hat{r}) = E(r)\, dr$.) We obtain:

$$\Delta V = -\int_b^a \frac{Q}{4\pi\varepsilon_0 r^2}\, dr = \frac{Q}{4\pi\varepsilon_0 a} - \frac{Q}{4\pi\varepsilon_0 b}$$

which yields a capacitance

$$C = \frac{Q}{\Delta V} = 4\pi\varepsilon_0 \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} = 4\pi\varepsilon_0 \frac{ab}{b-a}$$

Comment: If we let the distance $d = b - a$ between the spherical shells become very small compared to a and b, we may write

$$\frac{ab}{b-a} = \frac{ab}{d} \simeq \frac{a^2}{d}$$

so that

$$C \simeq \varepsilon_0 \frac{4\pi a^2}{d} = \varepsilon_0 \frac{A}{d}$$

where $A = 4\pi a^2 \simeq 4\pi b^2$ is the area of each of the spherical shells. I.e., the same result as for a parallel plate capacitor.

8) The forces due to the charges in B and C are equal but with opposite direction and give zero net contribution. The force from the charge in A points along OD. The vector sum of the

forces from the two charges at the middle of CD and BD must point along OA. In absolute value, each of these must be exactly twice as large as the force from the charge in A, since the distance OA is $\sqrt{2}$ times the distance from O to the two "midpoints". In absolute value, the vector sum of the forces from the two charges at the middle of CD and BD is a factor $\sqrt{2}$ times bigger than the length of each of them. Hence, this vector sum must be larger than the force from the charge in A. In conclusion, the total force must point along OA.

9) Opposite charges attract each other, charges of the same type repel each other. In addition, a charged object will always attract a neutral object because of polarization of the neutral object: For example, a positively charged object will push positive charges in the neutral object away and attract the negative ones. The net effect is attraction because of the shorter distance to the negative charges. (Example: Charge a comb by pulling it through your hair. The neutral water running in your sink will be attracted to the comb.)
So: Spheres 2 and 3 must both be charged, with charge of the same sign. Sphere 1 may be neutral or charged. If it is charged, the sign is the opposite of spheres 2 and 3. In any case, it will be attracted by sphere 2. This is all we can say, so we do not have enough information to determine what type of charge we have on all three spheres.

10) Newton's 3. law: If one object acts with a force on another object, the second object also acts on the first one, with a force of the same size, but opposite in direction.

11) Rather than calculating the field, we follow the hint and check whether any of the given alternatives reproduce well-known socalled "asymptotic" limits. For example: The field from a certain amount of localized charge Q in very large distance x must reduce to the field from a point charge Q in distance x, i.e.,

$$E(x) \to \frac{Q}{4\pi\varepsilon_0 x^2}$$

if $x \gg R$. For this particular problem, this is enough, because the expressions in B, C, and D don't even approach zero in this limit! (Besides, the expression in D doesn't have the correct dimension.) What about A, then, is it correct? Well, if $x \gg R$, we have

$$\begin{aligned} 1 - \frac{x}{\sqrt{x^2+R^2}} &= 1 - \frac{1}{\sqrt{1+R^2/x^2}} \\ &= 1 - \left(1 - \frac{R^2}{2x^2} + \ldots\right) \\ &\simeq \frac{R^2}{2x^2} \end{aligned}$$

so that $E \simeq Q/4\pi\varepsilon_0 x^2$. We could also go to the other limit, namely $x \to 0$. In that case, we should obtain the same as for an infinitely large plane, i.e., $\sigma/2\varepsilon_0$. And only the field in A is consistent with this:

$$E_A \to \frac{Q}{2\pi\varepsilon_0 R^2} = \frac{\sigma}{2\varepsilon_0}$$

with $\sigma = Q/\pi R^2$, i.e., charge pr unit area.

12) Since $E(r) = -dV/dr$, we must for the linear part of E have $E(r) = ar$ (a = positive constant), i.e., $dV/dr = -ar$. Hence, $V(r) = b - ar^2/2$ (b = constant), a parabola with negative curvature. Only curve 3 is appropriate. We see that the behavior of E and V is also OK for large values of r in curve 3, for example, $E(r) \sim 1/r^2$ and $V(r) \sim 1/r$.

13) Only charges *inside* the Gaussian surface contribute to the net electric flux through it. Field lines from charges outside will go both in and out through the Gaussian surface, and therefore do not contribute to net flux.

14) It does not matter if the charge distribution on the metal is no longer uniform: The electrostatic field $\boldsymbol{E}$ is everywhere zero inside the metal, so the potential V must be constant everywhere inside the metal. Do not forget this: In electrostatic equilibrium, a "connected" piece of electric conductor is an equipotential.

15) Here, we may use Gauss' law. We choose a Gaussian surface lying entirely within the sphere in the middle. Since we are then inside a conductor, we have $E = 0$ everywhere on the Gaussian surface, and hence $\oint \boldsymbol{E} \cdot d\boldsymbol{A} = 0$. According to Gauss' law, the net charge inside this surface is then zero. We may put this Gaussian surface arbitrarily close to the inner surface of the sphere in the middle. Since the charge on the inner sphere is $2Q$, the charge on the inner surface of the sphere in the middle must be $-2Q$. The sphere in the middle has total charge $-Q$. Inside the volume of this sphere, there can be no net charge (we proved this in the lectures, using Gauss' law). Hence, the rest of the charge, Q, must be on the outer surface of the sphere in the middle. (Based on a similar argument, we may also conclude that on the outer sphere, we have a charge $-Q$ on the inner surface and zero charge on the outer surface. So, outside all the three spheres, the electric field is ....)

16) $\boldsymbol{E} = -\nabla V$, i.e., $\boldsymbol{E}$ points in the direction of lower potential, and furthermore, is perpendicular to the equipotential surface.

17) The electric field strength inside the dielectric slab is reduced in comparison with having only vacuum there. Hence, the potential difference ΔV between the capacitor plates becomes smaller. A is therefore incorrect. Since the capacitance is given by $C = Q/\Delta V$, this means that C becomes larger when ΔV becomes smaller and Q is held fixed. B is therefore also wrong. The potential energy stored in the capacitor must have become smaller after we inserted the dielectric slab. For this conclusion, we may consider the energy pr unit volume in terms of the electric field, $u = \varepsilon_0 E^2/2$, or we might have calculated the work required to charge the capacitor from zero charge to final charge $\pm Q$. So, C is also incorrect. We are left with D, which is correct: The electric field in the air-filled layers is not affected by the dielectric slab. We have an induced negative charge on the upper surface and an equal amount of positive charge on the bottom surface of the dielectric slab. In the two layers of air, these two planes of induced charge give equal, but oppositely directed contributions to the total electric field.

18) We have e.g. $\boldsymbol{E} = \boldsymbol{F}/Q$ and $\Delta V = -\int \boldsymbol{E} \cdot d\boldsymbol{l}$. Besides, $C = Q/\Delta V$. So we may write down several possibilities:

$$[E] = [F/Q] = [\Delta V/l] = [F/C\Delta V]$$

Hence, these units are possible:

$[E] = \mathrm{N/C} = \mathrm{V/m} = \mathrm{N/FV}$.
The unit kg $\mathrm{m^2/s^2}$ C is not correct. (It would have been with m instead of m^2.)

19) If we are in the region between the inner sphere and the spherical shell, the net charge inside a spherical Gaussian surface is $2Q$. Hence, the electric field is

$$E(r) = \frac{2Q}{4\pi\varepsilon_0 r^2}$$

directed radially outwards. If we are outside the spherical shell, the net charge inside a spherical Gaussian surface is $2Q - Q = Q$. Hence, the electric field is

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

also here directed radially outwards. Figure 2 is therefore correct. We see that this is also consistent with the fact that field lines for E start on positive charges and end up on negative charges (or at infinity): The charge $-Q$ on the spherical shell must distribute itself with $-2Q$ on the inner and $+Q$ on the outer surface. (See question 15.) Everywhere inside the two conductors, we must of course have $E = 0$.
Figure 1 would have been correct if the metal shell had not been there at all. Figure 3 would have been correct if the metal shell had net charge $-4Q$ (and therefore charge $-2Q$ on both the inner and outer surface). Figure 4 would have been correct if the metal shell had net charge $-6Q$ (and therefore charge $-2Q$ on the inner surface and $-4Q$ on the outer surface).

20) An infinitely large plane with charge σ pr unit area creates an electric field $\sigma/2\varepsilon_0$. The electric force on another plane with charge Q put into this field is therefore $F = QE = Q\sigma/2\varepsilon_0$. The force pr unit area becomes $f = F/A = Q\sigma/2\varepsilon_0 A = \sigma^2/2\varepsilon_0$. With $\sigma = 10^{-5}$ $\mathrm{C/m^2}$ and $\varepsilon_0 = 8.85 \cdot 10^{-12}$ $\mathrm{C^2/N\ m^2}$, we have $f = 100/2 \cdot 8.85 \simeq 5.7$ $\mathrm{N/m^2}$.
Note that we cannot use the *total* electric field σ/ε_0 when we want to find the force on one plate from the other. The two plates each contribute with $\sigma/2\varepsilon_0$ to the total field, but any given one of them does not act on itself with a net force. (Just like you cannot lift yourself by the hair....!)