Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
Vår 2005

Solution to øving 9

Guidance week 9 and 10.

Question	A	B	C	D
1			x	
2		x		
3				x
4			x	
5				x
6	x			
7				x
8				x
9			x	
10	x			
11				x
12	x			
13			x	
14			x	
15				x
16			x	
17	x			
18				x
19	x			
20				x

1) The charge Q is distributed over the surface of the conductor and results in some kind of surface charge density σ (which will in general vary from place to place on the conductor's surface, unless it is spherical). The twice as large charge $2Q$ will distribute itself in the same manner and result in a twice as large surface charge density everywhere, 2σ. The electric field in point P can then be computed with Coulomb's law. With charge Q:

$$\boldsymbol{E}_1(P) = \int_S \frac{\sigma\, dA}{4\pi\varepsilon_0 r^2}\hat{r}$$

With charge $2Q$:

$$\boldsymbol{E}_2(P) = \int_S \frac{2\sigma\, dA}{4\pi\varepsilon_0 r^2}\hat{r} = 2\boldsymbol{E}_1(P)$$

Here, the integrals are taken over the (closed) surface S of the conductor, r is the distance from the surface element dA to P, and $\hat{r}$ is a unit vector along the direction from dA to P.

Since the field everywhere (i.e., everywhere *outside* the conductor!) has become twice as large, the potential (relative to zero, chosen at infinity)

$$V(P) = -\int_{\infty}^{P} \boldsymbol{E} \cdot d\boldsymbol{l}$$

must also have become twice as large.

2) Since $E = 0$ everywhere inside the metal sphere, the net charge Q must (because of Gauss' law) distribute itself over the surface of the sphere, and from symmetry reasons, uniformly over the surface. Then Gauss' law yields, with a spherical Gaussian surface with a radius r larger than the radius of the metal sphere,

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

i.e., as for a point charge Q at the center of the sphere. The potential $V(r)$ must therefore also become as for a point charge,

$$V(r) = \frac{Q}{4\pi\varepsilon_0 r}$$

Inside the metal sphere and on the surface of the sphere, the potential is constant and equal to $Q/4\pi\varepsilon_0 R$, where R is the radius of the sphere.
If the distance from the center of the sphere to B is twice the distance to A, the field is reduced by a factor of 4, while the potential is reduced by a factor of 2.

3) The potential on a metal sphere with radius R and charge Q is (see the former question)

$$V(R) = \frac{Q}{4\pi\varepsilon_0 R}$$

Let us e.g. put $V_0 = V(a) = Q/4\pi\varepsilon_0 a$, so that $V_1 = V_0/2 = 15V_0/30$, $V_2 = V_0/3 = 10V_0/30$ and $V_3 = V_0/5 = 6V_0/30$. Then we see that $V_1 : V_2 : V_3 = 15 : 10 : 6$.
It is intuitively clear that the potential is largest for the smallest sphere: Imagine starting with a neutral metal sphere and adding charge until the total charge is Q. Clearly, it must be harder to add this charge the smaller the sphere is, due to repulsion between the charges added. So, you must perform a larger amount of work in order to put the charge on the smaller sphere. In other words, the smallest sphere ends up with the largest potential energy, and therefore also the largest electric potential.
[Had the charge been negative, the smallest sphere would have had the *smallest* electric potential, i.e., the *most negative* potential relative to $V(\infty) = 0$. But still the smallest sphere would have had the largest potential energy, since $\Delta U = Q\Delta V = -|Q|\Delta V$ when $Q < 0$.]

4) Let us find the direction of the electric field in the vicinity of the two charges, knowing that the electric field points towards a negative charge. Just to the left of $x = -a$, the field must point right, i.e., $E(x) > 0$. All curves are consistent with this. But just to the right of $x = -a$, the field must point left, i.e., $E(x) < 0$. Curve 1 is not consistent with this. Just to the left of $x = a$, the field must point right, i.e., $E(x) > 0$. Curve 4 is not consistent with this. And finally, just to the right of $x = a$, the field must point left, i.e., $E(x) < 0$. Curve 2 is not

consistent with this. We are left with curve 3, which is correct. We see that $E(0) = 0$ in curve 3, which obviously must be correct.

5) $E = 0$ inside the metal sphere, thus A and C are out of the question. The field lines in B correspond, as we shall see in a few weeks, to the magnetic field around a current carrying wire perpendicular to the paper plane.

6) First, let's agree what happens here: The charge q is uniformly distributed on the surface of the metal sphere and creates an electric field $E_0(r) = q/4\pi\varepsilon_0 r^2$ outside (i.e. $r > R$; $E = 0$ inside the metal sphere). Electric dipoles in the plastic are aligned because of E_0, and the net effect of the polarization is an induced negative charge $-q_i$ on the inner surface of the plastic layer and a positive charge q_i on the outer surface of the plastic layer. The induced charge $-q_i$ creates an electric field $E_i(r) = -q_i/4\pi\varepsilon_0 r^2$ inside the plastic, directed opposite to E_0, i.e. radially inwards, so that the total electric field $E = E_0 + E_i$ becomes weaker in the plastic than if we had only air. On the outside of the plastic, the field strength is "reestablished" by the induced charge q_i on the outer surface. Alternatively, with Gauss' law: Total charge inside a spherical Gaussian surface with radius $r > 2R$ is $q - q_i + q_i = q$, so the electric field here is $E_0(r) = q/4\pi\varepsilon_0 r^2$. Total charge inside a spherical Gaussian surface with radius $R < r < 2R$ is $q - q_i$, so here, i.e., in the plastic layer, the electric field is $E(r) = (q - q_i)/4\pi\varepsilon_0 r^2$.
Polarization $\boldsymbol{P}$ is, by definition, electric dipole moment pr unit volume. We only have dipoles within the dielectric plastic layer, so P can be nonzero only here. Field lines for E would be as in the figure, but in addition we would have new field lines starting on the outer surface of the plastic layer.
We are left with electric displacement D, and we remember that D is determined by free charge: Gauss' law for D reads

$$\oint \boldsymbol{D} \cdot d\boldsymbol{A} = Q_{\text{free}}$$

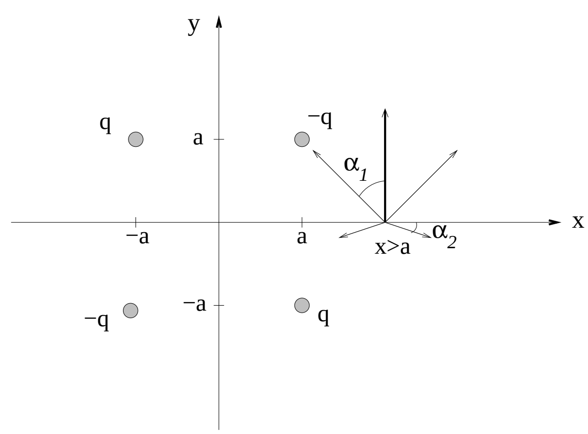
And by *free* charge, we simply mean all charge except induced (bound) charges associated with the polarization of dielectrics that are present in our system. Here, the charge q on the metal sphere represents the free charge, whereas $\pm q_i$ represent induced, bound charge.
Gauss' law for D immediately yields $D(r) = q/4\pi r^2$ everywhere outside the metal sphere, and the field lines are consistent with this.

7) Here, we may assume that we have approximately infinitely large planes. Then we know that the electric field from one plane equals the charge pr unit area on the plane divided by $2\varepsilon_0$, i.e., independent of the distance from the plane. Therefore, we may immediately exclude C, since the field outside must be zero (an equally large, but oppositely directed field contribution from the two planes when we are on the outside). Further, A must be wrong: There is no chance that the electric field is stronger inside the dielectric than in the layer with vacuum.
But why isn't B correct? Didn't we learn that the field would become weaker if we inserted a dielectric? Well, yes, but: Imagine starting with vacuum everywhere. Then the charges $\pm Q$ must be uniformly distributed on the two plates. Next, we insert the dielectric in the left half. Because of the field from the metal plates, the dipoles in the dielectric will be aligned with the external field. The net effect of this alignment is an induced surface charge on the dielectric, in our case positive on top and negative at the bottom. If nothing more than this happened, we would no longer have electrostatic equilibrium: It is no longer "advantageous"

to have the free charge on the metal plates uniformly distributed. The induced positive charge on top of the dielectric will pull some of the free electrons from the right half of the upper metal plate to the left half, and at the bottom, the opposite thing will take place. And when do we have electrostatic equilibrium? When the potential everywhere on the upper metal plate has the same constant value V_- and the potential everywhere on the lower metal plate has the same constant value V_+. (Remember: A metal plate is an equipotential in equilibrium!) In equilibrium, we have the same *total* charge density on the left and the right side. On the right side, we have $\sigma = \sigma_f^0$ and on the left side we have $\sigma = \sigma_f^1 - \sigma_i$. Here, subscript f denotes free charge on the metal plates and i denotes induced charge. With plate area A, these charge densities must of course fulfill the following: $\pm Q = \pm(\sigma_f^0 A/2 + \sigma_f^1 A/2)$ = total free charge on the metal plates. From symmetry reasons, the electric field must still be directed perpendicularly to the plates, and since the potential is constant on a given plate, the field must be uniform, $E = \Delta V/d = (V_+ - V_-)/d$, where d is the distance between the plates.

8) Zero field inside the metal eliminates A and C. Polarization in the plastic reduces the field with a factor $\varepsilon_1/\varepsilon_0 = 10$ in comparison to what we would have with vacuum or air present. The electric field decays as $1/r^2$, but this gives only a "reduction factor" of $(5/4)^2/(5/2)^2 = 1/4$ when we compare positions B and D. Thus, the field will be largest in position D.

9) In the point $(x, 0)$ $(x > a)$, the charges contribute with pairwise equally large field (in absolute value), and with directions as shown in the figure:



The vector sum of the four thin vectors results in a total electric field pointing in the positive y direction.

10) The potential from a point charge q is

$$V(r) = \frac{q}{4\pi\varepsilon_0 r}$$

where r is the distance from the point charge. The total potential is the sum of the potentials from each point charge (remember, the superposition principle). Here, we have pairs of charges with opposite sign but equal distance. Hence, the sum must be zero. Thinking a little, we realize that the whole xz plane is in fact an equipotential surface with potential $V = 0$. Obviously, the same must also be the case for the yz plane.

11) In the lectures, we used a parallel plate capacitor as an example and started with

$$U = \int_0^Q v(q)\, dq$$

and then showed that with an electric field $\boldsymbol{E}$, we have a potential energy pr unit volume

$$u(E) = \frac{1}{2}\varepsilon_0 E^2$$

Here, $v(q)$ represented the potential difference between the capacitor plates, so that $v(q)\, dq$ represented the work required to increase the charge on the capacitor plates from $\pm q$ to $\pm(q + dq)$. Hence, U becomes the total potential energy stored in the capacitor when we have charged it from 0 to final charge $\pm Q$.

This means that we have two alternative ways of determining the potential energy: We may associate U with the charge Q and use the first formula, of course provided we know $v(q)$. Alternatively, we may associate U with the electric field and use the formula

$$U = \int_V u(E)\, dV = \int_V \frac{1}{2}\varepsilon_0 E^2\, dV$$

provided we know what the field looks like in the volume V.

In this question, the system is quite simple, and we know both the potential of the metal sphere when it has a charge q,

$$v(q) = \frac{q}{4\pi\varepsilon_0 R}$$

and the electric field in all the space surrounding the sphere,

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}$$

when is is "fully" charged. Of course, it is sufficient to calculate the potential energy U one way, but let us do it both ways here, in order to check that we obtain the same answer.

U associated with the charge Q:

$$U = \int_0^Q v(q)\, dq = \int_0^Q \frac{q}{4\pi\varepsilon_0 R}\, dq = \frac{Q^2}{8\pi\varepsilon_0 R}$$

U associated with the field E:

$$U = \int_V u(E)\, dV = \frac{1}{2}\varepsilon_0 \int_R^\infty E^2 \cdot 4\pi r^2\, dr = \frac{1}{2}\varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0}\right)^2 \cdot 4\pi \int_R^\infty \frac{r^2\, dr}{r^4} = \frac{Q^2}{8\pi\varepsilon_0 R}$$

Here, we used spherical coordinates, but since E only depends upon r, we could do the two angular integrations "directly" and use the volume element $dV = 4\pi r^2\, dr$ = volume of a spherical shell with radius r and thickness dr.

We see that the answers are equal. In conclusion, we may choose if we like to associate the potential energy with the charges or with the electric field created by the charges.

12) The total charge on the two capacitors coupled in parallel is

$$Q = Q_1 + Q_2$$

Since $C_1 = Q_1/\Delta V$ and $C_2 = Q_2/\Delta V$, we may write

$$Q = C_1\Delta V + C_2\Delta V = (C_1 + C_2)\Delta V$$

The total capacitance is (pr definition)

$$C = \frac{Q}{\Delta V}$$

so we find

$$C = C_1 + C_2$$

In other words: The capacitance of two capacitors coupled in parallel equals the sum of the capacitances of each capacitor. Here, we have shown that this is true for two capacitors. It is straightforward to generalize to an arbitrary number of capacitors coupled in parallel:

$$C = \sum_i C_i$$

where the sum over i runs from 1 to N = the number of capacitors coupled in parallel.

13) The total voltage drop across the two capacitors coupled in series is

$$\Delta V = \Delta V_1 + \Delta V_2$$

Since $C_1 = Q/\Delta V_1$ and $C_2 = Q/\Delta V_2$, we may write

$$\Delta V = \frac{Q}{C_1} + \frac{Q}{C_2} = Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

The total capacitance is

$$C = \frac{Q}{\Delta V}$$

so we obtain

$$C = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)^{-1}$$

In other words: The inverse capacitance of two capacitors coupled in series equals the sum of the inverse capacitances of each capacitor. Here, we have shown that this is true with two capacitors. It is straightforward to generalize to an arbitrary number of capacitors:

$$\frac{1}{C} = \sum_i \frac{1}{C_i}$$

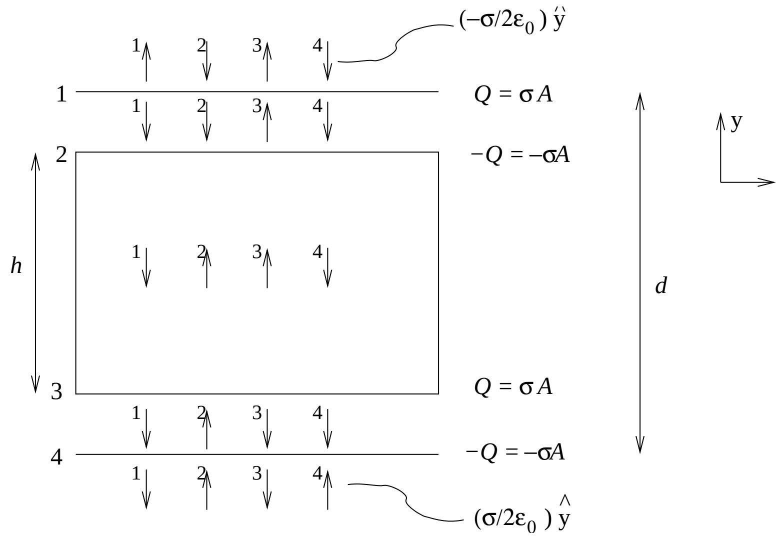
where the sum over i runs from 1 to N = the number of capacitors coupled in series.

14) The electric field from the two capacitor plates will induce charge on the upper and lower surface of the inserted metal sheet. In equilibrium, we must have zero electric field inside the

metal sheet. This is achieved if the induced charge is $-Q$ on the upper surface and Q on the lower surface of the metal sheet. Why exactly $-Q$ and Q? Well, because the electric field from an infinitely large charged plane is independent of the distance to the plane, and given by the surface charge density σ:

$$E_0 = \sigma/2\varepsilon_0$$

In our case, $\sigma = Q/A$, where A is the plate area. The direction of the field from a charged plane is *away from* the plane if it is positively charged and *towards* the plane if it is negatively charged. Our system then becomes as shown in the figure. With e.g. the y axis upwards, the different contributions to the total electric field in the different regions of space then are either $(-\sigma/2\varepsilon_0)\hat{y}$ or $(\sigma/2\varepsilon_0)\hat{y}$, see the figure.



So we have essentially 4 infinitely large planes, two with charge density σ (1 and 3) and two with charge density $-\sigma$ (2 and 4). In the figure, we have drawn the contributions from each of the four planes in all the five "different" regions. The total electric field is simply the vector sum in each region, so

$$E = 0$$

on the outside, and inside the metallic sheet. In the two regions between the metallic sheet and the capacitor plates we see that the field becomes

$$\boldsymbol{E} = -\frac{\sigma}{\varepsilon_0}\hat{y}$$

i.e., the same as before we inserted the metal sheet.
Finally, to what the question was all about, namely the potential difference between the capacitor plates. Without the metallic sheet present, the potential difference becomes

$$\Delta V = -\int_{(-)}^{(+)} \boldsymbol{E} \cdot d\boldsymbol{l} = \frac{\sigma d}{\varepsilon_0}$$

because then $\boldsymbol{E} = (-\sigma/\varepsilon_0)\hat{y}$ everywhere between the plates. (We choose $d\boldsymbol{l} = dy\,\hat{y}$.)
With the metal sheet present, the potential difference becomes

$$\Delta V = -\int_{(-)}^{(+)} \boldsymbol{E} \cdot d\boldsymbol{l} = \frac{\sigma(d-h)}{\varepsilon_0} = \frac{\sigma d}{3\varepsilon_0}$$

because now $\boldsymbol{E} = (-\sigma/\varepsilon_0)\hat{y}$ only in the two regions between the metal sheet and the capacitor plates. These two regions have a total extent $d - h = d - 2d/3 = d/3$; inside the metal slab we have $E = 0$.
In conclusion: The potential difference becomes three times smaller.
Comment: This was a very lengthy solution. Nevertheless, I personally find the approach suggested above a nice way of thinking in questions like these, with infinitely large charged planes. The only thing we need to know is that the field from one plane is $\sigma/2\varepsilon_0$, with direction away from or towards the plane, with positive or negative charge, respectively. In addition, we must of course know that $E = 0$ inside a metal. Finally, we need to know that the superposition principle applies for the electric field.

15) An infinitely large uniformly charged plane with charge pr unit area σ creates a constant electric field $E = \sigma/2\varepsilon_0$. If σ is positive, E is directed away from the plane, and opposite if σ is negative. Further, we know that the potential *decreases* if we move *with* the electric field. Remember: Decreasing potential when we move away from a positive charge, since a positive test charge must have decreasing potential energy when it moves away from a positive charge. Here, we have a positively charged plane, so the potential will decrease when we move away from the plane. Is has been chosen the value $V = -20$ V on the plane, so V must here be negative everywhere.
(If the plane had been negatively charged, with charge density $\sigma = -4$ nC/m^2, we would have had $V = 0$ in a distance $d = \Delta V/E = \Delta V \cdot 2\varepsilon_0/\sigma = 20 \cdot 2 \cdot 8.85 \cdot 10^{-12}/4 \cdot 10^{-9} \simeq 0.09$ m.)

16) If the electric field is E_0 between the metal plates before we insert the two dielectric layers, the field strength will be reduced by a factor $1/\varepsilon_{rj}$ inside dielectric number j ($j = 1, 2$). Here, ε_{rj} is the relative permittivity of dielectric j, i.e., $\varepsilon_{r1} = 4$ and $\varepsilon_{r2} = 2$. The field strength is not changed in the vacuum layer between $x = 2a$ and $x = 3a$. Hence:

$$\begin{aligned} E &= \frac{1}{4}E_0 \qquad 0 < x < 2a \\ E &= E_0 \qquad 2a < x < 3a \\ E &= \frac{1}{2}E_0 \qquad 3a < x < 5a \end{aligned}$$

This means that the potential will decrease most slowly between 0 and $2a$, fastest (more precisely: four times faster) between $2a$ and $3a$, and "intermediately" fast (more precisely: twice as fast as between 0 and $2a$) between $3a$ and $5a$. Only curve 3 is consistent with this.

17) See question 7: There, we agreed that the electric field $\boldsymbol{E}$ must be constant throughout the space between the two plates. Next, we must have $\boldsymbol{P} = 0$ in the right half: No electric dipoles to be aligned in vacuum! However, in the dielectric we have such electric dipoles that are aligned, resulting in a polarization $\boldsymbol{P}$ with the same direction as $\boldsymbol{E}$. But then it should be clear that the electric displacement $\boldsymbol{D} \equiv \varepsilon_0\boldsymbol{E} + \boldsymbol{P}$ becomes larger in the left half. This is also consistent with what we said, that $\boldsymbol{D}$ can be associated with free charge: In question 7, we concluded that we had the largest amount of free charge on the left half of the metal plates, where we have the dielectric.

18) Starting from questions 7 and 17, we have agreed that the electric field is constant in the

whole region between the plates, and also that the density of free charge on the metal plates is biggest on the side where we have the dielectric present. Hence, we may write

$$E_1 = \frac{\sigma^{(1)}}{\varepsilon_0} = \frac{\sigma_f^{(1)} - \sigma_i^{(1)}}{\varepsilon_0}$$

for the field in region 1, to the left, and

$$E_2 = \frac{\sigma^{(2)}}{\varepsilon_0} = \frac{\sigma_f^{(2)}}{\varepsilon_0}$$

for the field in region 2, to the right. Here, $\sigma^{(1)}$ and $\sigma^{(2)}$ are *total* charge densities on the left and right side, respectively, $\sigma_f^{(1)}$ and $\sigma_f^{(2)}$ are *free* charge density (on the metal plates) on the left and right side, respectively, and $\sigma_i^{(1)}$ is *induced* charge density (on the surface of the dielectric) on the left side. These fields are supposed to be equal, $E_1 = E_2 = E$, and the potential difference between the plates is determined by this field strength:

$$\Delta V = Ed$$

Let us briefly repeat the various physical quantities that we introduced in the lectures when we talked about polarization in linear media. We assumed that the polarization is proportional to the electric field:

$$\boldsymbol{P} = \chi_e \varepsilon_0 \boldsymbol{E}$$

where $\chi_e =$ the *susceptibility* of the medium. With the *definition*

$$\boldsymbol{D} \equiv \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$$

we could then write

$$\begin{aligned} \boldsymbol{D} &= \varepsilon_0 \boldsymbol{E} + \boldsymbol{P} \\ &= (1 + \chi_e)\varepsilon_0 \boldsymbol{E} \\ &= \varepsilon_r \varepsilon_0 \boldsymbol{E} \\ &= \varepsilon \boldsymbol{E} \end{aligned}$$

where we had introduced $\varepsilon_r \equiv 1 + \chi_e =$ the *relative* permittivity and $\varepsilon \equiv \varepsilon_r \varepsilon_0 =$ the permittivity of the medium.
The electric dipole moment of the dielectric in region 1 is

$$p_1 = (\sigma_i^{(1)} A/2)d$$

so the polarization here becomes

$$P_1 = \frac{p_1}{(Ad)/2} = \sigma_i^{(1)}$$

since the volume of region 1 is $Ad/2$. Then, the electric displacement in region 1 becomes

$$D_1 = \varepsilon_0 E_1 + P_1 = \sigma_f^{(1)} - \sigma_i^{(1)} + \sigma_i^{(1)} = \sigma_f^{(1)}$$

In region 2, we have $P_2 = 0$, so that

$$D_2 = \varepsilon_0 E_2 = \sigma_f^{(2)}$$

At the same time, we have

$$D_1 = \varepsilon E_1$$

Let us look at what the total charge on the metal plates is:

$$\begin{aligned}
Q &= \frac{A}{2}\left(\sigma_f^{(1)} + \sigma_f^{(2)}\right) \\
&= \frac{A}{2}(D_1 + D_2) \\
&= \frac{A}{2}(\varepsilon E_1 + \varepsilon_0 E_2) \\
&= \frac{A}{2}E(\varepsilon + \varepsilon_0) \\
&= \frac{A}{2}\frac{\Delta V}{d}\varepsilon_0(\varepsilon_r + 1)
\end{aligned}$$

Pr definition, the capacitance of the capacitor is

$$C = \frac{Q}{\Delta V}$$

so here we have

$$C = \frac{A}{2d}\varepsilon_0(\varepsilon_r + 1) = \frac{\varepsilon_r + 1}{2}C_0$$

with $C_0 = \varepsilon_0 A/d$.
It should perhaps not come as a big surprise that this capacitor can be viewed as two capacitors coupled in parallel, both with plate area $A/2$, plate distance d, and one of them filled with vacuum and the other filled with a dielectric with permittivity $\varepsilon = \varepsilon_r\varepsilon_0$. Thus, we could actually have written down the result more or less directly, using the result in question 12.

19) This question is similar to the previous one, except that we have here effectively two capacitors coupled in series instead of parallel. Let us try the simple solution first, equipped with the wisdom learnt in question 18. The two capacitances, coupled in series both have plate area A and distance between the plates $d/2$. One of them is filled with air/vacuum and the other is filled with a dielectric with permittivity $\varepsilon = \varepsilon_r\varepsilon_0$. Then we may use the result found in question 13. The capacitance of the half filled with dielectric is

$$C_1 = \varepsilon\frac{A}{d/2} = 2\varepsilon_r\varepsilon_0\frac{A}{d}$$

whereas the capacitance of the half filled with air is

$$C_2 = \varepsilon_0\frac{A}{d/2} = 2\varepsilon_0\frac{A}{d}$$

Total capacitance is then, according to question 13

$$\begin{aligned} C &= \left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1} \\ &= \varepsilon_0\frac{A}{d}\left(\frac{1}{2\varepsilon_r}+\frac{1}{2}\right)^{-1} \\ &= \varepsilon_0\frac{A}{d}\frac{2\varepsilon_r}{\varepsilon_r+1} \\ &= \frac{2\varepsilon_r}{\varepsilon_r+1}C_0 \end{aligned}$$

Also in this question, we may go the long way via electric displacement, induced and free charge etc., and determine the relation between total free charge Q and the potential difference ΔV. We have:

$$D = \sigma_f = Q/A$$

In this case, D is constant throughout the region between the plates since the free charge is uniformly distributed over the metal plates (no difference between right and left here). Next, we have

$$D = \varepsilon E_1$$

for the relation between electric displacement and electric field in region 1 (i.e., the lower half with dielectric). Furthermore,

$$D = \varepsilon_0 E_2$$

for the relation between electric displacement and electric field in region 2 (i.e., the upper half, with vacuum). The potential difference between the plates is

$$\Delta V = E_1\cdot\frac{d}{2}+E_2\cdot\frac{d}{2}$$

which is found by taking the line integral of $\boldsymbol{E}$ from one of the plates to the other. But now we have more or less what we need:

$$\begin{aligned} \Delta V &= E_1\cdot\frac{d}{2}+E_2\cdot\frac{d}{2} \\ &= \frac{d}{2}\left(\frac{D}{\varepsilon}+\frac{D}{\varepsilon_0}\right) \\ &= \frac{d}{2\varepsilon_0}\left(\frac{Q}{A\varepsilon_r}+\frac{Q}{A}\right) \\ &= \frac{Qd}{2\varepsilon_0 A}\cdot\frac{1+\varepsilon_r}{\varepsilon_r} \end{aligned}$$

so that

$$C = \frac{Q}{\Delta V} = \frac{2\varepsilon_0 A}{d}\cdot\frac{\varepsilon_r}{1+\varepsilon_r} = \frac{2\varepsilon_r}{1+\varepsilon_r}C_0$$

In other words, the same as found above, by simply using the formula for two capacitances coupled in series!

20) Here, the electric field in the region between the inner and outer metal cylinder is given in the text, so it's just a question of calculating the potential difference:

$$\begin{aligned}\Delta V &= V_a - V_b = -\int_b^a E(r)\, dr \\ &= \frac{\lambda}{2\pi\varepsilon}\int_a^b \frac{dr}{r} \\ &= \frac{\lambda}{2\pi\varepsilon}(\ln b - \ln a) \\ &= \frac{Q}{2\pi\varepsilon L}\ln\frac{b}{a}\end{aligned}$$

Hence, the capacitance of the cylindrical capacitor becomes

$$C = \frac{Q}{\Delta V} = \frac{2\pi\varepsilon L}{\ln b/a}$$

The biggest problem here is perhaps keeping track of the overall sign, and for example end up with alternative C. However, then you must remember: A capacitance is, pr definition, a *positive* quantity. Since $a < b$, alternative D simply must be the correct one, because the logarithm of a number smaller than 1 is negative.
Another thing to note, perhaps, is that we will have the high value of the electric potential at the positive conductor. This follows from the definition of electric potential, in terms of potential energy pr unit charge. Imagine a small positive charge. It must obviously have the largest potential energy if we choose to put it in a position near the positively charged conductor. Hence, here is also the highest value of the electric potential. For a small negative charge, it is opposite: It will have the highest potential energy if we choose to put it in a position near the negatively charged conductor. So, here we must have the lowest value of the electric potential, so that when we multiply the value of the potential with the negative value of the charge, we end up with (the largest) positive value of the potential energy.

One final comment: Do you think this exercise required a lot of work? I agree. The midterm exam will contain 40 questions. Obviously, this cannot be 40 questions that require as much work as some of the questions above. This applies in particular to some of the last questions, which really required hard work, much because it's new and unfamiliar topics. On the other hand: Questions like nr 5 should not take too long time. Either you know that $E = 0$ inside a metal sphere in equilibrium, or you don't!
With 40 questions and 3 hours, you will have 4.5 minutes for each question. Since all correctly answered questions count equally, it may be a good idea to do the "quick ones" first.