Institutt for fysikk, NTNU
TFY4155/FY1003 Elektrisitet og magnetisme
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Solution to øving 15

Guidance Thursday April 28 and Friday April 29

Exercise 1

a) Examples:
For an infinitely large charged plane with charge σ pr unit area, the electric field is oppositely directed on the two sides, and the field strength is $\sigma/2\varepsilon_0$. Hence, a *discontinuity* of σ/ε_0 in the normal component of the field, and no discontinuity in the tangential component of the field.
Inside a metal sphere with charge Q, the electric field is zero. On the surface, at $r = R$, the field is $Q/4\pi\varepsilon_0 R^2$, directed radially outwards (if $Q > 0$), i.e., normal to the surface. Again, a discontinuity of σ/ε_0 in the normal component of the field, since $\sigma = Q/4\pi R^2$ is the charge pr unit area on the surface of the sphere.
An infinitely large plane carrying a uniform current i pr unit length results in a uniform magnetic field $\mu_0 i/2$, in opposite direction on the two sides of the plane, see øving 13, exercise 2. I.e., a discontinuity of $\mu_0 i$, and if you take a look at øving 13, you will find out that we're talking about the component of $\boldsymbol{B}$ that lies in the plane of the current, and at the same time is normal to the current direction.
Inside an infinitely long solenoid, the magnetic field is $\mu_0 nI$, outside it is zero. Hence, a discontinuity of $\mu_0 nI$. The current is I pr turn, while the number of turns pr unit length is n, so $i = nI$ becomes the current pr unit length. Again, a discontinuity of $\mu_0 i$.
Comment: No boundary surface is infinitely large, but if we come sufficiently close to the surface, it will look as if it is infinite and flat. The total electric field on the surface must be equal to the sum of the contributions form the "nearby region", i.e., the part of the surface that looks large and flat, and the contribution from all the charges in "the rest of the world". The charges in the rest of the world are all far away from the "crossing point", i.e., far away when compared to the charges that are actually *in* the plane where we are crossing. "The rest of the world" therefore must contribute with the same field just below and just above the boundary surface, i.e., with a contribution that is *continuous*. In other words, the whole discontinuity in the electric field is due to the charges in the plane that we cross. And correspondingly for the magnetic field: Total magnetic field is the sum of the contributions from the current *in* the plane where we cross, and the contribution from all other currents in the world. Only the current in the plane where we cross contribute to the discontinuity.

b) Dielectric slab perpendicular to constant external electric field $\boldsymbol{E}_0$:
Here we have boundary surfaces that are perpendicular to the fields. We cannot use the boundary condition for $\boldsymbol{E}$ because we do not know how much charge we have in the boundary surfaces between vacuum and the dielectric. We know that there is an induced (bound) charge, positive in the upper surface and negative in the lower surface, but not *how much*. However, we can use the boundary condition for $\boldsymbol{D}$ because we know that there is zero *free* charge in the slab. Hence, we must have $D_1 = D_0$, where $D_0 = \varepsilon_0 E_0$ is the electric displacement outside the

slab. In addition, we have $D_1 = \varepsilon_1 E_1 = \varepsilon_r \varepsilon_0 E_1$. Thus:

$$\begin{aligned} D_1 &= \varepsilon_0 E_0 \\ E_1 &= \frac{1}{\varepsilon_r} E_0 \end{aligned}$$

Dielectric slab parallel to constant external electric field $\boldsymbol{E}_0$:
Now we have boundaries parallel to the field direction. Then we can use that the parallel component of $\boldsymbol{E}$ is continuous, i.e., $E_1 = E_0$. The relation $D_1 = \varepsilon_r \varepsilon_0 E_1$ is of course still valid, so

$$\begin{aligned} D_1 &= \varepsilon_r \varepsilon_0 E_0 \\ E_1 &= E_0 \end{aligned}$$

Magnetizable slab perpendicular to constant external magnetic field $\boldsymbol{B}_0$:
Here we have again boundaries perpendicular to the fields. We may therefore use the fact that B_n is continuous, i.e., $B_1 = B_0$. In addition we have $B_1 = \mu_1 H_1 = \mu_r \mu_0 H_1$. Hence

$$\begin{aligned} H_1 &= \frac{1}{\mu_r \mu_0} B_0 \\ B_1 &= B_0 \end{aligned}$$

Magnetizable slab parallel to constant external magnetic field $\boldsymbol{B}_0$:
Boundaries are now parallel to the fields. We know that there is an induced magnetization current in the surface of the slab, but not how much. However, we may use the boundary condition for $\boldsymbol{H}$, because we know that there is zero free current in the slab. Hence we have $H_1 = H_0$, where $H_0 = B_0/\mu_0$ is the H field outside the slab (vacuum). In addition we have $B_1 = \mu_r \mu_0 H_1$. Thus

$$\begin{aligned} H_1 &= \frac{1}{\mu_0} B_0 \\ B_1 &= \mu_r B_0 \end{aligned}$$

Explanation of different E_1 and B_1 in the two situations:
Dielectric slab perpendicular to external field results in polarization, and a corresponding induced charge in the surface. The induced charge contributes with an electric field opposite to the external field, so that E_1 becomes smaller than E_0. With the slab parallel to the external field, the induced surface charge on the slab is localized infinitely far away from "where we are". Therefore, it does not contribute anything to the field "where we are", and $E_1 = E_0$.
A magnetizable slab parallel to the external field results in magnetization, and a corresponding induced current in the surface of the slab. The induced current contributes to the field in the same direction as the external field, so that B_1 becomes larger than B_0. With the slab perpendicular to the external field, the induced surface current is localized infinitely far away from "where we are". Hence, it does not contribute anything to the field "where we are", and $B_1 = B_0$.

Exercise 2

The *external* current I generates an H field $H = nI$ along the solenoid everywhere inside the solenoid (because of Ampere's law for H.) So, we can simply use that

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M}\right) = \mu_r \mu_0 \boldsymbol{H}$$

to determine the various quantitites:

Inside the iron:

$$H_j = nI = 2000 \text{ m}^{-1} \cdot 3 \text{ A} = 6000 \text{ A/m}$$
$$B_j = \mu_r \mu_0 H_j = 2000 \cdot 4\pi \cdot 10^{-7} \text{ (Vs/Am)} \cdot 6000 \text{ A/m } = 15 \text{ T}$$
$$M_j = (\mu_r - 1) H_j = 1.2 \cdot 10^7 \text{ A/m}$$

In the airfilled part inside the solenoid:

$$H_0 = H_j = 6000 \text{ A/m}$$

$$B_0 = \mu_0 H_0 = 7.5 \text{ mT}$$
$$M_0 = 0$$

The calculated value of the magnetization inside the iron rod, $M_j = 1.2 \cdot 10^7$ A/m, is larger than the saturation magnetization $M_s = 1.6 \cdot 10^6$ A/m, and therefore not possible. The reason is that we have used the linear relation $B = \mu_r \mu_0 H$ between the magnetic field B and the field H from the external current. However, here we have such a strong external field H that this linear relation is no longer valid. All magnetic dipoles are already aligned with the external field when $H \simeq M_s/\mu_r = 800$ A/m. An additional increase in H cannot raise the value of M any further.
Corrected, maximum value of B_j becomes

$$B_j^{\text{korr}} = \mu_0 \left(H_j + M_s\right) = 4\pi \cdot 10^{-7} \cdot \left(6000 + 1.6 \cdot 10^6\right) = 2 \text{ T}$$

Exercise 3

a) The area enclosed by the loop is $L \cdot x$, increasing linearly with time, $dA/dt = L dx/dt = Lv$. Enclosed magnetic flux is $\phi = B \cdot A = BLx$, so induced emf in the loop becomes $\mathcal{E} = d\phi/dt = BL dx/dt = BLv$. The loop has resistance R, so by Ohm's law the current is

$$I = \mathcal{E}/R = BLv/R$$

The current runs counterclockwise. This is because the force on a positive charge q in the metal rod, $q\boldsymbol{v} \times \boldsymbol{B}$, is directed upwards. Altenratively, with Lenz' law: Increasing area gives increasing enclosed magnetic flux into the plane. The current must then run in a direction so that the flux due to I points out of the plane when we are inside the loop.

b) The current I runs upwards in the metal rod, the magnetic field points into the plane, so the magnetic force on the current I is directed to the left, and is

$$F = ILB = B^2L^2v/R$$

c) From the previous point, we have the result that we must pull the rod to the right with a force F in order to keep constant velocity. If we release the rod, the only force acting on it is the (breaking) magnetic force to the left. Newton's 2. law then gives us the following equation of motion:

$$m\frac{dv}{dt} = -\frac{B^2L^2v}{R}$$

This is a 1. order differential equation for $v(t)$. Solution:

$$\begin{aligned}
\frac{dv}{v} &= -\frac{B^2L^2}{mR}dt \\
\Rightarrow \ln v &= -\frac{B^2L^2}{mR}t + \ln k \\
\Rightarrow v(t) &= k\, e^{-B^2L^2t/mR} \\
\Rightarrow v(t) &= v_0\, e^{-B^2L^2t/mR}
\end{aligned}$$

where we used the initial condition $v(0) = v_0$.

d) We must calculate how much energy that is lost as heat in the resistor. The power loss is VI, where V is the voltage drop across the resistor, i.e., $V = RI$. Since power loss is energy loss pr unit time, we may write

$$dW = P\,dt = VI\,dt = RI^2\,dt$$

for the energy dW lost in a time dt. We have, in *a* above, found I in terms of the velocity v, so it's simply a matter of inserting into the equation. The total energy loss must be the integral of dW, i.e., from $t = 0$ to $t = \infty$:

$$\begin{aligned}
W &= \int dW \\
&= \int_0^\infty RI^2\,dt \\
&= \int_0^\infty R\left(\frac{BLv}{R}\right)^2 dt \\
&= \frac{B^2L^2v_0^2}{R}\int_0^\infty e^{-2B^2L^2t/mR}\,dt \\
&= \frac{B^2L^2v_0^2}{R}\Bigg|_0^\infty \left(-\frac{mR}{2B^2L^2}\right) e^{-2B^2L^2t/mR} \\
&= mv_0^2/2
\end{aligned}$$

which we were supposed to show.

Exercise 4

a) Magnetic field from long, straight current carrying wire is $B(x) = \mu_0 I/2\pi x$, where x is the distance from the wire. On the plane which is enclosed by the quadratic loop, the magnetic field points out of the paper plane. The x axis is chosen upwards. The magnetic field and the "surface element vector" $d\boldsymbol{A}$ are parallel, so we simply have $d\phi = B \cdot dA$ for the flux through a surface element dA. Here, we choose a horizontal stripe of width a and height dx as our surface element. The total flux enclosed by the quadratic loop is obtained by integrating, from $x = d$ to $x = d + a$. (We have chosen $x = 0$ in the position of the wire.)

$$\begin{aligned}
\phi &= \int d\phi \\
&= \int_d^{d+a} \frac{\mu_0 I}{2\pi x} a\, dx \\
&= \frac{\mu_0 I a}{2\pi} \Big|_d^{d+a} \ln x \\
&= \frac{\mu_0 I a}{2\pi} \ln \frac{d+a}{d}
\end{aligned}$$

b) Induced emf equals the time derivative of the enclosed magnetic flux:

$$\begin{aligned}
\mathcal{E} &= -\frac{d\phi}{dt} \\
&= -\frac{\mu_0 I a}{2\pi} \frac{d}{dt} (\ln(d+a) - \ln d) \\
&= -\frac{\mu_0 I a}{2\pi} \left(\frac{1}{d+a} \frac{dd}{dt} - \frac{1}{d} \frac{dd}{dt} \right) \\
&= \frac{\mu_0 I a v}{2\pi} \frac{a}{(d+a)d}
\end{aligned}$$

Here we have used $v = dd/dt$, and now d is no longer a constant distance, but a distance that increases linearly with time, e.g., $d(t) = d_0 + vt$.
The flux out of the plane decreases with time. Hence, the induced emf and the corresponding current runs *counterclockwise*, in order to oppose this change. (Again, Lenz' law.)

c) If the loop is pulled horizontally, the enclosed flux does not change. Hence, the induced emf is zero.