Institutt for fysikk, NTNU
TFY4155/FY1003: Elektrisitet og magnetisme
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Summary, week 6 (February 8 and 9)

Meaning of the gradient operator

The vector ∇V points in the direction where V increases most rapidly, i.e., in the direction where the *directional derivative* of V has its largest value. Since $\boldsymbol{E} = -\nabla V$, this means that the electric field points in the direction where V *decreases* most rapidly.
Example: If a point charge q is placed in a position where $\nabla V = 0$, it is not subject to any forces, because $\boldsymbol{F} = q\boldsymbol{E} = -q\nabla V = 0$.

Summary so far, and saying hello to Maxwell equation nr 1

The Coulomb law (empirical law for force between two charges q and q' in mutual distance r):

$$\boldsymbol{F} = \frac{qq'}{4\pi\varepsilon_0 r^2}\hat{r}$$

Electric field from point charge q (follows from the definition "force pr unit charge"):

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q'} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

Conservative force:

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{l}$$

is independent of the integration path, i.e., the path between the points A and B. Hence:

$$\oint \boldsymbol{F} \cdot d\boldsymbol{l} = 0$$

(i.e., when we integrate around a *closed* path)

With the definition of $\boldsymbol{E}$, it follows that the electrostatic field is also conservative, i.e.:

$$\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l}$$

is independent of the integration path, and hence

$$\oint \boldsymbol{E} \cdot d\boldsymbol{l} = 0$$

This is one of *Maxwell's equations* (for static fields, i.e., fields that do not change in time).
A conservative vector field can always be derived from a scalar *potential*:

$$\boldsymbol{E} = -\nabla V$$

The potential difference between two points A and B can be evaluated if we know the electric field in the space between A and B:

$$\Delta V = V_B - V_A = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l}$$

The superposition principle is valid for the elektric force $\boldsymbol{F}$ (this is an experimental result):

$$\boldsymbol{F}_i = \sum_{j=1}^{n} \boldsymbol{F}_{ij}$$

= force on charge q_i from charges q_j $(j = 1, 2, \ldots n)$
Then it follows that the superposition principle is also valid for the electric field $\boldsymbol{E}$,

$$\boldsymbol{E} = \sum_{j=1}^{n} \boldsymbol{E}_j$$

and for the electric potential V,

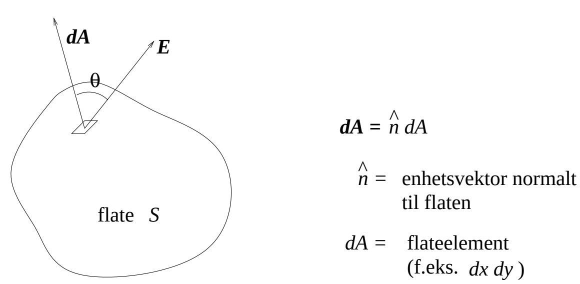
$$V = \sum_{j=1}^{n} V_j$$

Here, $\boldsymbol{E}_j$ and V_j are the contributions to the field and the potential, respectively, form charge number j.

Electric flux
[FGT 23.1; YF 22.1; TM 22.2; AF 25.3; LHL 19.7; DJG 2.2.1]

$$\phi = \int_S \boldsymbol{E} \cdot d\boldsymbol{A}$$

Sometimes, we write ϕ_E to specify that it's an *electric* flux we're talking about. Earlier, we have defined electric field lines so that the electric field strength $E = |\boldsymbol{E}|$ is proportional with the density of field lines, i.e., the number of field lines pr unit area. From the above definition of electric flux ϕ, we may conclude that ϕ simply represents the number of field lines that cross the surface S.
The following figure illustrates what this is about:



The surface S is an arbitrary "thought" or "chosen" surface in space. The electric field "exists" in the region where the surface S is "located". ($\boldsymbol{E}$ may be zero, or nonzero.) The surface S is then divided into small surface elements $d\boldsymbol{A} = \hat{n} dA$, with area dA and an orientation in space specified by the *surface normal* $\hat{n}$. The flux $d\phi$ through the surface dA is then $\boldsymbol{E} \cdot d\boldsymbol{A}$. The total flux through the whole surface S is obtained by integrating the contributions $d\phi$, which is the equation above.

Note that the flux is a *scalar* quantity. However, it may be positive or negative, depending on whether the angle between the vectors $\boldsymbol{E}$ and $d\boldsymbol{A}$ is smaller or larger than 90 degrees.

A *closed surface* S is a surface which encloses a well defined volume V, e.g., a spherical shell, a peanut shell or similar. The electric flux through a closed surface is written like this (cf. the notation used for path integral around a closed curve):

$$\phi_c = \oint_S \boldsymbol{E} \cdot d\boldsymbol{A}$$

The index c denotes "closed". It is actually not necessary as long as we write down the integral sign with the ring on it. The latter is sufficient to make sure we're talking about a closed surface.
With a closed surface, we may introduce a sign convention for the surface element vector: We choose *positive* direction for $d\boldsymbol{A}$ when it is directed *out of* the surface.
Then we may conclude that

$$\begin{aligned} \boldsymbol{E} \cdot d\boldsymbol{A} > 0 &\Rightarrow \text{flux out through thesurface} \\ \boldsymbol{E} \cdot d\boldsymbol{A} < 0 &\Rightarrow \text{flux in through thesurface} \end{aligned}$$

Furthermore:

$$\begin{aligned} \phi_c > 0 &\Rightarrow \text{net flux out through thesurface} \\ \phi_c < 0 &\Rightarrow \text{net flux in through thesurface} \end{aligned}$$

For a surface S that is *not closed*, we don't have this opportunity to choose the positive and the negative direction for $d\boldsymbol{A}$. The surface has two sides, and none of these can be claimed to be more "inside" than the other. However, we may solve the problem by choosing a positive direction of the (closed!) curve that runs around the edge of S. Then, the positive direction of $d\boldsymbol{A}$ is chosen in terms of the right hand rule: Let the four fingers of your right hand point along the positive direction of the curve around S. Then, the remaining finger, the thumb, points in the positive direction of $d\boldsymbol{A}$.

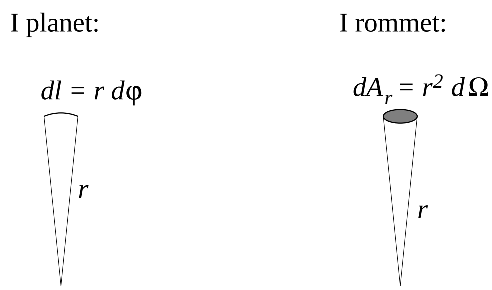
Gauss' law
[FGT 23.2; YF 22.3; TM 22.2, 22.6; AF 25.4; LHL 19.7; DJG 2.2.1]

Gauss' law (in socalled integral form; later, if time permits, we shall see that we also have a version of Gauss' law on socalled differential form):

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{A} = \frac{q_{\text{in}}}{\varepsilon_0}$$

Here, the integral on the left hand side denotes a surface integral over a closed surface S, while q_{in} is the total *net* charge *inside* this closed surface (the "gaussian surface).
Gauss' law is one of *Maxwell's equations.* (For the moment, we are only talking about *electrostatics.* However, Gauss' law is also valid when the electric field changes in time.)
The content in Gauss' law may be formulated like this: The net number of field lines out of a volume, i.e., out through the closed surface enclosing this volume, is determined by, and is directly proportional with, the net charge inside this volue, i.e., inside the closed surface.
Gauss' law follows directly from Coulomb's law, and therefore really represents no new physics.

In connection with the proof of Gauss' law, we used a thing we called a *solid angle* Ω. In just the same way as a small sector in a plane spans an *angle* $d\phi$, a small sector in space (3D) spans a *solid angle* $d\Omega$. Furthermore: In the same way as the arc length dl in distance r from the "origin" then becomes $dl = r\ d\phi$, the area dA_r of the surface which is perpendicular to $\boldsymbol{r}$ and limited by the small sector then becomes $dA_r = r^2\ d\Omega$. Note the analogy between 2D and 3D!



If we let the sector in the plane go once around, this corresponds to the angle

$$\oint d\phi = \int_0^{2\pi} d\phi = 2\pi$$

Analogously: If we let the sector in space span the whole sphere, this corresponds to a solid angle

$$\oint d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta = 4\pi$$

(Here, we have used spherical coordinates, where $dA_r = r^2 \sin\theta\ d\theta\ d\phi$ (see øving 3!), so that $d\Omega = dA_r/r^2 = \sin\theta\ d\theta\ d\phi$.)