AMERICAN JOURNAL of PHYSICS

The missing wave momentum mystery

David R. Rowland and Colin Pask

Citation: Am. J. Phys. **67**, 378 (1999); doi: 10.1119/1.19272
View online: http://dx.doi.org/10.1119/1.19272
View Table of Contents: http://ajp.aapt.org/resource/1/AJPIAS/v67/i5
Published by the American Association of Physics Teachers

The missing wave momentum mystery

David R. Rowland and Colin Pask
School of Mathematics and Statistics, University College, The University of New South Wales, Australian Defence Force Academy, Canberra ACT 2600, Australia

(Received 9 July 1998; accepted 18 September 1998)

The usual suggestion for the longitudinally propagating momentum carried by a transverse wave on a string is shown to lead to paradoxes. Numerical simulations provide clues for resolving these paradoxes. The usual formula for wave momentum should be changed by a factor of 2 and the involvement of the cogenerated longitudinal waves is shown to be of crucial importance.

I. INTRODUCTION

This paper deals with two key elements in physics. First, the conservation of energy and momentum principles—probably our two most important principles for understanding and analysing physical phenomena. Second, the propagation of waves along a stretched string—the canonical example used when studying continuum systems and wave theory.

First recall that the special theory of relativity tells us that energy has associated with it a relativistic mass, and so propagating energy, be it particle or wave, has associated with it a relativistic momentum.[1] The classic example of a wave carrying momentum is of course the electromagnetic field, even though the wave itself is made up of zero rest mass particles. The question as to whether all forms of propagating energy carry *nonrelativistic* momentum though, is not so straightforward, however, with Pierce[2] and Juenker[3] for example warning the reader to be wary. In fact, Rayleigh showed in 1905 that in fluids of certain hypothetical pressure–density behavior, a wave might carry zero or even negative momentum.[4] It is also well known in solid state physics, that while a phonon in a crystal might carry the pseudomomentum $\hbar\mathbf{k}$, it carries no real physical momentum.[5–9]

Transverse waves travelling down a taut string are a well-studied phenomenon,[2,3,10–16] with the first full analytic solution of the vibrating string being given by Lagrange back in 1759. Consequently, one might expect that the question of what actual momentum, if any, such waves carry in their direction of motion would have been satisfactorily answered. We show in this paper, however, that previously presented answers to this question lead to paradoxes and so must either be incorrect, or at least incomplete. We resolve these paradoxes using numerical simulations as a guide to the development of a complete and fully self-consistent theory.

The paper is arranged as follows. In Secs. II and III we describe our basic stretched-string model and discuss the relevant definitions of energy and momentum. Then in Sec. IV, we show that the standard model and definitions lead to a mystery expressed as paradoxes arising when wave reflections and tension in the string are analyzed. In Sec. V, an examination of a conservation principle and its origins provides a partial resolution to the first paradox. We then pause for an important aside—the questions we raise are really quite general and we discuss longitudinal waves in a rod as an example. Using numerical simulations as a guide, the paradoxes are resolved in Sec. VII and corrections to standard presentations of wave theory are presented. This is followed by some concluding remarks and appendices containing a few more technical details.

Before we proceed, we should also mention that we were led to consider this problem from our research into conservation laws for modal interactions in multimoded nonlinear optical waveguides.[17,18] Conservation of momentum would be expected to give one such conservation law for such problems. The topic of electromagnetic momentum in material media, however, is a difficult one fraught with controversy.[9,19,20] To clarify some of the issues, we decided to follow Shockley's advice of ''try simplest cases,''[21] mechanical oscillations on strings or in rods being presumably simpler models in which to investigate the physics of wave momentum.

II. PHYSICAL MODEL

We take as our physical model the standard ideal string[10,11] which is assumed to be perfectly flexible and linearly elastic. By perfectly flexible it is meant that the string has no flexural rigidity and so the only restoring force acting on string elements is a tensile force acting everywhere tangential to the local string direction. Linear elasticity, on the other hand, implies that the tensile force is assumed to depend linearly on the amount the string is stretched from its undeformed length.[10]

With the above assumptions, we can model our string as a linear chain of point masses joined by ideal (massless and Hookean) springs, which are stretched from their relaxed or unstressed length a (see Fig. 1). The equations of motion for the mass m_j are thus

$$m_j\ddot{x}_j=-k\left(1-\frac{a}{l_{j,j-1}}\right)(x_j-x_{j-1})+k\left(1-\frac{a}{l_{j,j+1}}\right)\times(x_{j+1}-x_j), \tag{1a}$$

$$m_j\ddot{y}_j=-k\left(1-\frac{a}{l_{j,j-1}}\right)(y_j-y_{j-1})+k\left(1-\frac{a}{l_{j,j+1}}\right)\times(y_{j+1}-y_j), \tag{1b}$$

$$l_{j,j\pm1}=[(x_{j\pm1}-x_j)^2+(y_{j\pm1}-y_j)^2]^{1/2}, \tag{1c}$$

where k is the spring constant of each of the springs; x_j is the longitudinal and y_j the transverse coordinate of the mass m_j; and $l_{j,j\pm1}$ is the distance between the masses m_j and $m_{j\pm1}$, respectively. It is important to note that when no waves are present, each spring has a length l greater than its relaxed length a so that a string under tension is modelled. (This

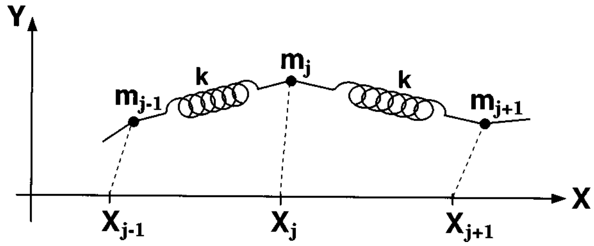

Fig. 1. Schematic of our model of an ideal string: a lattice of point masses joined by massless, linearly elastic springs with spring constants k. As a wave passes, the point masses m_j move from their equilibrium positions $(X_j,0)$ to $(x_j(t),y_j(t))$. In its undisturbed state, $|X_{j\pm 1}-X_j|=l>a$, where a is the relaxed length of each spring, so that a string under tension is modelled.

allows longitudinal waves to propagate in a flexible solid like a string—longitudinal waves in this case being waves of changes in local tension—because the separation between masses never falls below a.)

Taking the usual continuum limit with these assumptions and to lowest order,[10,22] (see Appendix B) transverse wave motion on the string is determined by the linear wave equation

$$\rho_0 \frac{\partial^2 \eta}{\partial t^2} = \tau_0 \frac{\partial^2 \eta}{\partial x^2}, \tag{2}$$

where $\eta(x,t)\equiv y(x,t)$ gives the transverse displacement of the infinitesimal piece of string at position x as a function of time t, ρ_0 is the equilibrium linear mass density (mass per unit length) of the string, and τ_0 its undisturbed tension. Equation (2) is the usual textbook equation and the wave velocity is given by

$$c_{\mathrm{T}} = \sqrt{\frac{\tau_0}{\rho_0}}, \tag{3}$$

where the subscript T refers to ''transverse.'' The discrete and continuum models are linked through the parameters $\rho_0 = m/l$ and $\tau_0 = k(l-a)$. Note also that the derivation of Eq. (2) requires that $|\partial \eta/\partial x| \ll 1$.

III. ENERGY AND MOMENTUM CARRIED BY THE WAVE

In this section, we present the standard expressions for the energy and momentum densities carried by transverse waves on a string. In Sec. IV, we show that the expression for wave momentum density leads to paradoxes and so must be incorrect (at least in its interpretation).

Kinetic and potential energy is carried by the wave, with the kinetic energy part being due to the motion of mass elements $\delta m = \rho_0 \delta x$ and the potential energy part being a result of the work done in stretching the string against the (assumed constant) tension force. The total energy density (energy per unit length) ϵ carried by the wave is readily shown to be[10]

$$\epsilon = \frac{1}{2}\rho_0 \left(\frac{\partial \eta}{\partial t}\right)^2 + \frac{1}{2}\tau_0 \left(\frac{\partial \eta}{\partial x}\right)^2 \tag{4}$$

to lowest order. In the discrete model, energy is just the sum of the classical dynamics kinetic and potential energies for all particles and springs.

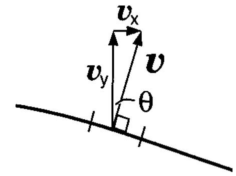

Fig. 2. Geometry of an element of string moving perpendicularly to its local direction. $\mathbf{v}$ is the actual velocity vector of the string element in this case, and it has components $\mathbf{v}_x$ and $\mathbf{v}_y$ in the x and y directions, respectively. The angle between $\mathbf{v}_y$ and $\mathbf{v}$ is θ.

For the discrete model, the definition of momentum is straightforward: Particle j has momentum with components $m\dot{x}_j$ and $m\dot{y}_j$ in the x and y directions (the overdot represents a time derivative). In the continuum case, there have been proposed several routes to an expression for the wave momentum density. One route is to observe that pure transverse motion of the string doesn't lead to momentum in the direction of propagation and so the assumption of pure transversality of string motion has to be dropped. Following this observation, several authors have argued that the string has to stretch in such a way that the instantaneous velocity of an infinitesimal segment of string is always perpendicular to the segment,[12–14] as shown in Fig. 2. If we assume this approximation to be valid (*and we show later that it isn't!*) then doing a bit of geometry we find that $\tan\theta = -\partial\eta/\partial x$, and since $|\partial\eta/\partial x| \ll 1$, we have $\theta \simeq -\partial\eta/\partial x$. Consequently, v_y $(\equiv \partial\eta/\partial t) = v\cos\theta \simeq v$ and hence $v_x = v\sin\theta \simeq -(\partial\eta/\partial t)\times(\partial\eta/\partial x)$. Thus the momentum density g (we use lower case letters for densities and upper case letters for totals) carried by the wave in the x direction is given by

$$g = \rho_0 v_x \simeq -\rho_0 \frac{\partial \eta}{\partial t}\frac{\partial \eta}{\partial x}. \tag{5}$$

[Note that we can neglect this x motion of string elements when calculating the energy density ϵ in Eq. (4), because it would only lead to a second-order correction (in the small quantity $\partial\eta/\partial x$) to the kinetic energy density term in ϵ.]

An *apparently* more rigorous derivation of this result is given by Elmore and Heald (Sec. 1.11 in Ref. 10), who make a detailed analysis of the forces acting on an element of string including the effects of stretching.

A third approach is outlined in Appendix A, where we show that the canonical energy–momentum tensor constructed from the Lagrangian density of the problem leads to the identification of a wave momentum density given by the formula for g above. In one sense, this is a remarkable agreement. The Lagrangian density leads directly to Eq. (2) and contains no description of the longitudinal motion of segments of the string.

With all this agreement on the formula for g, what could be wrong?

IV. ''THE MYSTERY''

A. Parodox 1: Nonconservation of wave momentum

The first paradox arises from a study of a transverse wave travelling down a taut string which has a discontinuity in its linear mass density at $x=0$ such that $\rho_0=\rho_1$ for $x<0$ and $\rho_0=\rho_2$ for $x\geqslant 0$ (see Fig. 3). This is a well-studied problem,[2,10–16] but one that can still surprise with its subtle-

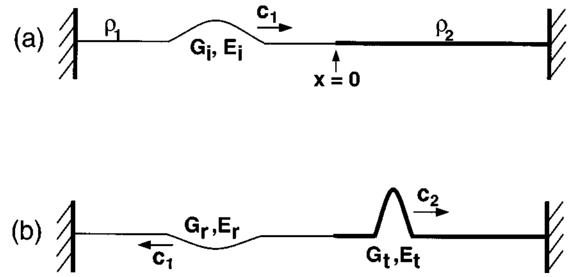

Fig. 3. Two strings with mass densities ρ_1 and ρ_2, respectively, are joined at $x=0$ and put under the tension τ_0. (a) A right-propagating wave pulse travelling with wave speed $c_1=\sqrt{\tau_0/\rho_1}$ is initially excited on the left string segment. The total energy and total longitudinal momentum of this wave pulse are E_i and G_i, respectively. (b) After reaching $x=0$, the incident wave pulse generates reflected (total energy E_r and total momentum G_r) and transmitted (total energy E_t and total momentum G_t) wave pulses.

ties as we will show. (We note in passing that all the following results also apply to compressional waves propagating along rods.)

The surprise came when we looked at trying to calculate the amplitudes of the reflected and transmitted waves from a consideration of energy and momentum conservation. We know from mechanics that the outcome of an elastic collision between a mass M_1, initially travelling with velocity v_{1i}, with an initially stationary mass M_2, is completely determined in a one-dimensional interaction by conservation of energy and momentum. Waves, like particles, (supposedly) carry energy and momentum, and one would expect (naively as it turns out) that conservation of energy and momentum would also help us determine the results of wave interactions. Specifically, one would expect that the total energy and momentum carried by the transmitted and reflected waves would equal the energy and momentum of the wave incident on the discontinuity. That is, from Fig. 3, that $E_r+E_t=E_i$ and $G_r+G_t=G_i$, where G is the total momentum carried by a wave pulse and E is the total energy carried by a wave pulse. The subscripts $\mu=i$, r, and t refer to incident, reflected, and transmitted waves, respectively.

Our results hold for the general case, but for definiteness, consider incident, reflected, and transmitted waves given by

$$\eta_i(x,t)=A_i e^{-(x-c_1t)^2/w^2}, \quad x<0, \tag{6a}$$

$$\eta_r(x,t)=A_r e^{-(x+c_1t)^2/w^2}, \quad x<0, \tag{6b}$$

$$\eta_t(x,t)=A_t e^{-(c_1^2/c_2^2)(x-c_2t)^2/w^2}, \quad x>0, \tag{6c}$$

respectively, where $t=0$ has been chosen to be the time when the peak of the incident wave is at $x=0$. [The standard requirement imposed when deriving Eq. (2), i.e., that $|\partial\eta/\partial x|\ll 1$, is thus equivalent to $|A_\mu/w|\ll 1$.]

The total energy and momentum of each of the three waves is found just by integrating the densities given by Eqs. (4) and (5) over the whole pulse at fixed times when the pulse is well away from $x=0$. Thus

$$E_i=\int_{-\infty}^{0}\epsilon_i\,dx\simeq\int_{-\infty}^{\infty}\epsilon_i\,dx, \quad t\ll -w/c_1, \tag{7a}$$

$$G_i=\int_{-\infty}^{0}g_i\,dx\simeq\int_{-\infty}^{\infty}g_i\,dx, \quad t\ll -w/c_1, \tag{7b}$$

with similar expressions for E_r, G_r, E_t, and G_t. Substituting Eqs. (6) into these integrals and using the fact that $\tau_0=\rho_1c_1^2=\rho_2c_2^2$, we find that

$$E_i=4\rho_1c_1^2IA_i^2/w, \tag{8a}$$

$$G_i=4\rho_1c_1IA_i^2/w, \tag{8b}$$

$$E_r=4\rho_1c_1^2IA_r^2/w, \tag{8c}$$

$$G_r=-4\rho_1c_1IA_r^2/w, \tag{8d}$$

$$E_t=4\rho_2c_1c_2IA_t^2/w, \tag{8e}$$

$$G_t=4\rho_2c_1IA_t^2/w, \tag{8f}$$

where $I=\int_{-\infty}^{\infty}u^2\exp(-2u^2)du=\sqrt{\pi}/2^{5/2}$. (In passing we note that $|G_\mu|=E_\mu/c_\mu$, $\mu=i,r,t$, a general result linking energy, momentum, and wave speed for linear waves.[1,2,10])

Applying conservation of energy, $E_i=E_r+E_t$, and momentum, $G_i=G_r+G_t$, to Eqs. (8), we find that

$$\frac{A_t^2}{A_i^2}=\frac{2c_2^2}{c_1(c_1+c_2)}, \tag{9a}$$

$$\frac{A_r^2}{A_i^2}=\frac{c_1-c_2}{c_1+c_2} \tag{9b}$$

with the unfortunate result that if $c_2>c_1$, then A_r is purely imaginary—a nonsensical result!

What went wrong? Well the usual way to find the reflection and transmission coefficients is to apply the boundary conditions that $\eta(x,t)$ and $\partial\eta/\partial x$ be continuous at $x=0$. This leads to[10,11]

$$\frac{A_r}{A_i}=\frac{c_2-c_1}{c_1+c_2}, \tag{10a}$$

$$\frac{A_t}{A_i}=\frac{2c_2}{c_1+c_2}. \tag{10b}$$

These results used in Eqs. (8a), (8c), and (8e) indicate that $E_r+E_t=E_i$, i.e., that energy is conserved in the process. However, the momentum equations [(8b), (8d), and (8f)] give

$$G_r+G_t=G_i\left[1+\frac{2(c_1-c_2)}{(c_1+c_2)}\right]. \tag{11}$$

Consequently, wave momentum is not conserved!! Herein lies the "mystery" of the title. If $c_2>c_1$, we apparently lose some momentum somewhere and if $c_2<c_1$ then we gain some momentum from somewhere. It is important to realise that Eq. (11) indicates a dramatic failure in the conservation of momentum law and not just some small error, associated with the small $|\partial\eta/\partial x|$ assumption for example.

B. Paradox 2: Differential tension in string

If the longitudinal motion of an element of string is governed by Eq. (5), then the total longitudinal displacement $\Delta\xi$ of an element of the string is given by

$$\Delta\xi=\int_{\text{wave}}\frac{\partial\xi}{\partial t}\,dt=\int_{\text{wave}}-\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}\,dt. \tag{12}$$

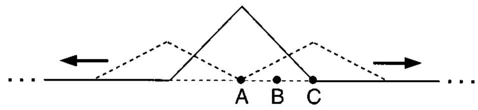

Fig. 4. A taut string is initially displaced transversally from equilibrium in the shape of a triangular wave (the solid curve) and released from rest. This results in waves with the same shape but half the amplitude travelling in both the positive and negative x directions (dashed curves).

For a wave of the form given by Eq. (6a), $\Delta\xi = 4A^2 I/w$, while for an identical wave travelling in the negative x direction, $\Delta\xi$ is the negative of this.

Consider now the case of a uniform long string plucked locally at its centre and released from rest as shown in Fig. 4. As is well known from d'Alembert's formula, this generates two waves of the same shape but with half the amplitude of the original displacement, one propagating in the positive and one propagating in the negative x directions. Infinitesimal elements of the string which are initially displaced will, however, see different fractions of each wave and so will be moved different fractions of the total displacement $\Delta\xi$ caused by the passing of the whole of the wave. For example, the centre element at the point A shown in Fig. 4 will see half of each wave, and so will receive a net displacement of zero. Elements to the left of this will receive increasingly greater negative displacements while elements to the right will receive increasingly greater positive displacements. For example, the element at point B in Fig. 4 will see ''one-quarter'' of the left-travelling wave and ''three-quarters'' of the right-travelling wave and so will be shifted ''half'' $\Delta\xi$ to the right. Point C, seeing none of the left-travelling wave and all of the right-travelling wave, will, however, be shifted all of $\Delta\xi$ to the right.

The above arguments lead to the conclusion that elements of the string will be left in a state of differential tension after the passage of the waves. This is just not physically possible (except maybe for a hypothetical string with zero Young's modulus) and so represents another mystery to be resolved.

V. PARTIAL RESOLUTION OF PARADOX 1

A. Continuity equations

The first mystery is concerned with the conservation or otherwise of wave energy and momentum, so it makes sense to determine the relevant mathematical descriptions of the conservation of these quantities.

In general terms, a physical quantity is conserved if the time rate of change of that quantity in an arbitrary volume in space equals the negative of the flux of the quantity through the boundary surface of that volume. For physical quantities that can be considered to be continuously distributed, the preceding statement can, with the help of the divergence theorem, be written as a triple integral over the volume in question.[22] Taking the limit as the volume goes to zero, we obtain a differential equation, a so-called continuity equation, which describes conservation of the physical quantity. The general form of continuity equations, in one dimension, is

$$\frac{\partial}{\partial t}(\text{density of physical quantity}) + \frac{\partial}{\partial x}(\text{current density of physical quantity}) = 0, \quad (13)$$

where in our case, the two physical densities of interest are energy ϵ and momentum g.

In order to get completely general results, we consider Eqs. (2)–(5) with ρ_0 replaced by $\rho(x)$, an arbitrary function of x.

Following the procedure developed for the case of ρ a constant, the continuity equation for energy when ρ is a function of x is found by multiplying the wave equation, Eq. (2), by $\partial\eta/\partial t$ and then reexpressing the result in the form of Eq. (13).[7,10,23] The resulting continuity equation is given by

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial P}{\partial x} = 0, \quad (14)$$

where ϵ is given by Eq. (4) [with ρ_0 replaced by $\rho(x)$] and the wave power, P, (energy current or flux density) is given by

$$P = -\tau_0 \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}. \quad (15)$$

We thus see that wave energy is conserved on a string with varying mass density. Comparing this result with Eq. (5) and using Eq. (3) leads to $P = c_T^2 g$, indicating the close link between momentum and energy.

For momentum, on the other hand, the continuity equation when ρ is a constant has been shown to be obtainable by multiplying the wave equation by $\partial\eta/\partial x$ and then reexpressing the result in the form of Eq. (13).[7,10,23] When $\rho = \rho(x)$, however, this procedure leads to

$$\frac{\partial g}{\partial t} + \frac{\partial b}{\partial x} = f(x), \quad (16)$$

where g is given by Eq. (5); and the wave momentum flow, b (momentum current or flux density) is, for linear waves, identical to the energy density ϵ. This equation is *not* a continuity equation however, because of the presence of the ''force density'' term f which is given by

$$f(x) = \frac{1}{2}\frac{d\rho}{dx}\left(\frac{\partial \eta}{\partial t}\right)^2. \quad (17)$$

From Eqs. (16) and (17), we thus see that *wave momentum is only conserved when the string mass density ρ is independent of position x*. (More generally, any variability in the string properties, such as τ for the case of a string hanging under its own weight,[11] will lead to nonconservation of wave momentum. In fact, for the case $\tau = \tau(x)$, wave energy isn't conserved either.)

That the term on the right-hand side of Eq. (16) can in fact be interpreted as a force density acting on the wave can be seen as follows. Integrating Eq. (16) on the string segment from $x = x_1$ to $x = x_2$, we find that

$$\frac{dG}{dt} = b(x_1) - b(x_2) + \int_{x_1}^{x_2} f(x)\,dx, \quad (18)$$

which has the ''physical'' interpretation: The time rate of change of wave momentum on $[x_1, x_2]$ equals the momentum flux in at $x = x_1$ minus the flux of momentum out at $x = x_2$ plus the total force acting on the wave in the interval. We shall leave a more detailed discussion of the nature of this ''force'' to Sec. VII.

B. Force on wave at the string discontinuity

Going back to our original problem of a step discontinuity in the mass density of the string at $x=0$, we model that case as follows. Letting

$$\rho(x)=\rho_1+(\rho_2-\rho_1)\frac{x}{\Delta x} \tag{19}$$

on the interval $[0,\Delta x]$ and taking the limit as $\Delta x\to 0$, we find that

$$\frac{d\rho}{dx}\to(\rho_2-\rho_1)\,\delta(x), \tag{20}$$

where $\delta(x)$ is the Dirac delta function.

Thus the total force F acting on the wave at any time t is given by

$$F=\int_{-\infty}^{\infty}f(x)\,dx=\frac{1}{2}\,(\rho_2-\rho_1)\left(\frac{\partial\eta(0,t)}{\partial t}\right)^2. \tag{21}$$

Using Eqs. (6), (8), and (10), the total change in momentum given to the wave by this force F is therefore

$$\Delta G=\int_{-\infty}^{\infty}F\,dt=\frac{2(c_1-c_2)}{(c_1+c_2)}\,G_i, \tag{22}$$

which is precisely the change in total wave momentum required by Eq. (11)!

Well that's a relief, but it begs the following question. If there is a force which acts on the wave at the discontinuity, there must be a reaction force by the wave on whatever it is that is applying the force to the wave. What is this thing that the wave applies a force to? Lots of thought and calculation reveals that it is nothing included in the current model, which is hardly surprising as this is a longitudinally acting ''force'' and some models, such as those beginning with a Lagrangian, have no longitudinal effects built into them. So the mystery remains.

The resolution of this additional mystery, and paradox 2 as well, will be given in Sec. VII.

VI. AN IMPORTANT ASIDE—LONGITUDINAL WAVES IN A ROD

The astute reader may have noted at this point that the equations describing longitudinal waves in rods are just the same as those for transverse waves on a string (with the Young's modulus Y times the cross-sectional area S replacing the tension τ, c_T becoming $c_L=\sqrt{SY/\rho}$, and the transverse displacement η being replaced by the longitudinal displacement ξ) and so may be wondering if there is a similar problem for those sorts of waves as well. It turns out that there isn't a problem, but there are some misleading discussions in the literature.

For longitudinal waves, both the continuity equation and Lagrangian field theoretic approaches (see Appendix A) lead to a wave momentum density given by Eq. (5) (with η replaced by ξ). Discussions in classic texts on mechanics such as Goldstein[22] and Corben and Stehle,[24] interpret the wave momentum density as being due to an ''excess density'' $\rho_e=-\rho_0\partial\xi/\partial x$ moving with the local displacement velocity $\partial\xi/\partial t$, correctly noting that for waves which (possibly periodically) return the particles of the medium to their equilibrium positions, the momentum density $\rho_0\partial\xi/\partial t$, when integrated over the wave, is identically zero. After reading the discussions in these texts, one is left believing that G $(=\int g\,dx)$ is in fact a real net momentum carried by the wave. Gilbert and Mollow,[6] however, show that G *isn't* a real momentum, as do other authors,[7–9] who also show that G is the classical limit of phonon pseudomomentum, and phonons are well known to carry no real momentum.[5] Numerical simulations of the kind discussed in Sec. VII verify that for waves like Eq. (6) (with η replaced by ξ), the total momentum carried by the wave as a whole is in fact zero, but that the pseudomomentum G is related to the wave energy E by the formula $G=E/c_L$, as predicted by the theory for wave momentum. *Thus G is a real property of such waves, it just isn't the real momentum carried by the wave.*

Since G is a pseudomomentum, its nonconservation on reflection and transmission at a discontinuity in the mass density of the supporting medium is, therefore, not a cause for any concern, just a cause to be careful in interpreting what is going on. Note also, that since G is a pseudomomentum, the force density found in Eq. (17) should be more properly called a pseudoforce, and so the fact that there is no reaction pseudoforce is therefore of no concern.

Of what interest is the wave pseudomomentum G, then? First, as Gilbert and Mollow[6] point out, it is, unlike real momentum, a nonzero integral of the motion. They also go on to show that if the wave is coupled to an external particle, then G *behaves* like a real momentum in the sense that G plus the momentum of the external particle is a conserved quantity, and using G rather than the centre of mass momentum may be more convenient in calculations, as is the case when calculating the results of neutron–phonon scattering for example (remembering that phonons also only have a pseudomomentum). Gurevich and Thellung also discuss the value of the conservation of pseudomomentum (or quasimomentum as they call it) in a homogeneous medium in both a nonlinear theory of elasticity[7] and a nonlinear theory of the interaction of light with matter.[8] Thus wave pseudomomentum is of considerable importance in the study of longitudinal waves.

As a final note of some importance for Sec. VII, we ask the question: ''Can a longitudinal wave in a solid ever carry a net real momentum?'' The answer is ''yes''—but the wave is not of the conventional type in which the medium is left unchanged after the wave passes. (For the discrete case of masses connected by springs, a conventional wave returns each mass to its original position.) For a longitudinal wave to carry a real net momentum, it must be set up so that the passage of the wave leaves each mass displaced from its original position.[3,7,25] Such a wave carries a real net momentum because it involves the transfer of mass. In this case, the momentum carried by the wave is still not given by G, but rather by $\int_{\mathrm{wave}}\rho_0\dot{\xi}\,dx$.

For this special ''mass-transferring wave'' case, the reflection-transmission problem for longitudinal waves can indeed be solved using conservation of energy and momentum. The interested reader is referred to Section 7.3 in the book by Ingard.[25]

VII. NUMERICAL SIMULATIONS—AND A COMPLETE RESOLUTION OF THE MYSTERY

We now report on the results of the numerical simulations and see how they lead us to the resolution of the paradoxes presented above and the correct formulation of wave theory for this problem.

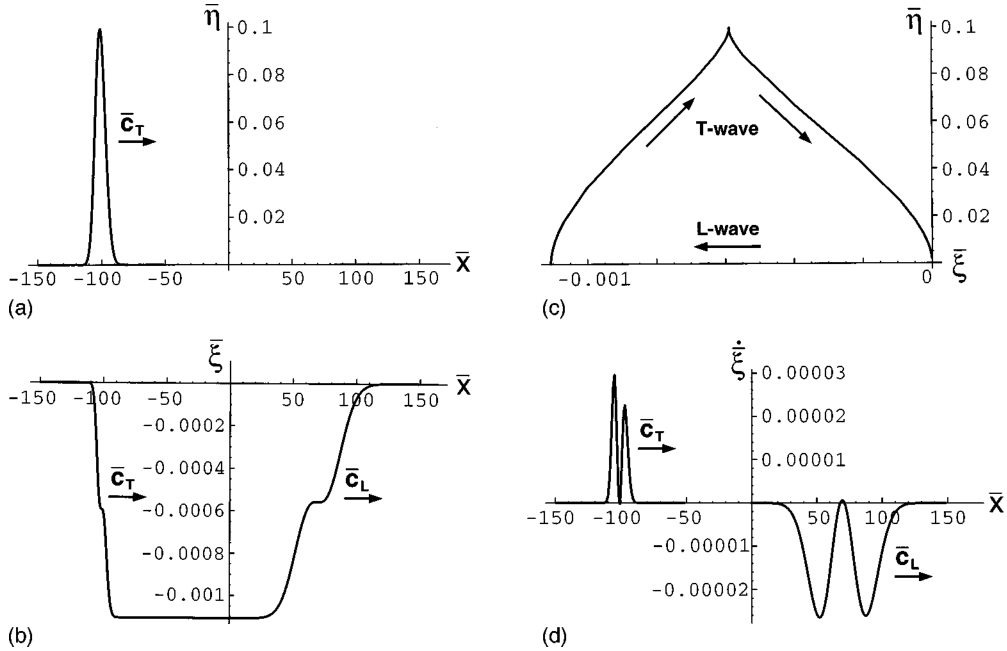

Fig. 5. Simulations for a string with $a/l=0.95$ and the normalizations used are defined at the start of Sec. VII A. The equilibrium positions of the point masses making up the lattice are at integer values of $\overline{X}$ on the $\overline{X}$ axis. The right end of the lattice is held fixed in place while the left end is given the displacements: $\overline{\xi}(-150,\overline{t})\equiv 0$, $\overline{\eta}(-150,\overline{t})=0.1(H(\overline{t})-H(\overline{t}-120))\exp(-((\overline{t}-60)/25)^2)$ in normalized units. In this expression, $H(\overline{t})$ is the Heaviside step function. (a) Plot of the transverse displacement $\overline{\eta}(\overline{X},280)$ at time $\overline{t}=280$, showing that a transverse Gaussian wave pulse has been set up on the string travelling at the normalized wave speed $\overline{c_T}=\sqrt{1-a/l}\simeq 0.2236$, as predicted by the standard theory. (b) A plot of the longitudinal displacements $\overline{\xi}(\overline{X},280)$ of each mass in the lattice at the same time as in (a). Observe that there are two longitudinal waves, a fast purely longitudinal precursor wave travelling at the normalised wave speed $\overline{c_L}=1$, and a slower longitudinal wave copropagating with the transverse wave. (c) Motion of the mass at $\overline{X}=-110$ showing the normalized transverse vs longitudinal displacements of this mass. The L wave arrives first and shifts the mass from (0,0) to $(-0.0011, 0)$ and then when the T wave arrives, it shifts the mass from $(-0.0011, 0)$ to $(-0.000\,59, 0.1)$ and thence back to (0,0). This mass was chosen so that the L and T waves had completely separated by this value of $\overline{X}$ (for earlier values of $\overline{X}$ where the L wave and T wave are still partially coincident, the motion of masses are more complicated). (d) Plot of normalized longitudinal momentum density $\dot{\overline{\xi}}\equiv\partial\overline{\xi}/\partial\overline{t}$ as a function of $\overline{X}$ at time $\overline{t}=280$. The graph of $-(1/2)(\partial\overline{\eta}/\partial\overline{X})(\partial\overline{\eta}/\partial\overline{t})$ is indistinguishable from the graph of $\dot{\overline{\xi}}$ for the T wave.

A. Simulation results

The numerical simulations were run on a HP server using MATHEMATICA[26] 3.0 to solve the coupled differential equations given in Eqs. (1) for N particles and suitable boundary conditions. The equations were first normalized so that distances are scaled relative to l, the equilibrium spacing between the masses when the string is under a tension τ_0; and time relative to $\sqrt{m/k}$, where m is the standard mass. With these normalizations, the mass density of a segment of string with standard masses is one unit, and is r units on a segment of string where $m_j=rm$. The normalized longitudinal wave velocity on a section of string with standard masses is thus $\overline{c_L}=1$, and $1/\sqrt{r}$ on a section of string with mass density r. The normalized tension in the string is $\overline{\tau_0}=(1-a/l)$,[27] and so the normalized transverse wave velocity is given by $\overline{c_T}=\sqrt{(1-a/l)/r}$.

As our first example, we considered setting up a transverse wave on a uniform spring–mass lattice under tension by moving the left-hand end vertically up and down in a smooth fashion (a Gaussian in time) with the right-hand end of the lattice held fixed in place. As can be seen in Fig. 5(a), a Gaussian transverse wave is set up which propagates as expected at the standard speed c_T for such waves.

However, the surprise comes when we plot the accompanying longitudinal displacement of the masses in the lattice as shown in Fig. 5(b). The results indicate that preceding the transverse wave is a longitudinal wave travelling with the speed $c_L=\sqrt{(ka+\tau_0)/\rho_0}$. For the example given in Fig. 5, this precursor longitudinal wave displaces all the masses in the lattice -0.0011 units to the left as it propagates.

Figure 5(b) also shows that there is a longitudinal motion copropagating with the transverse wave. This longitudinal motion is such as to return all the masses back to their equilibrium positions. The combined effects on a single particle of the precursor longitudinal wave (the "L wave") and the trailing (mostly) transverse "T wave" are shown in Fig. 5(c).

Now looking at the longitudinal momentum density profile shown in Fig. 5(d), we see that the precursor longitudinal wave is carrying a negative momentum, whereas the transverse wave via its accompanying longitudinal component is carrying a positive momentum in the x direction. Both of the longitudinal waves do carry a real momentum because they move mass as discussed at the end of Sec. VI. (We note here that precursor longitudinal waves have been observed in struck piano strings—see Refs. 28 and 29 and references therein—though these workers have not been interested in the momentum carried by these waves.)

Comparing the momentum density of the "transverse" wave with that predicted by the standard formula given by Eq. (5), we find that the numerically determined momentum density as a function of position is precisely one-half the g of Eq. (5)!![30] This refines a statement by Juenker that the momentum carried by a wave depends on the relative admixture in the wave of what he calls shape and density waves. We would say rather that the total wave consists of a part which is purely longitudinal (the L wave) and which propagates with the wave speed c_L, and a part (the T wave) that is a mixture of both transverse and longitudinal displacements but which is mostly transverse and travels at the wave speed c_T. The T wave carries a total longitudinal momentum K_T/c_T (K_T is the total kinetic energy of the transverse motion) while the L wave carries a total momentum which depends on how the waves are generated and on the parameter a/l. (By total momentum and kinetic energy, we mean the total sum of the momenta or kinetic energies of all the particles involved in each wave in the discrete model, which corresponds to the integral over the wave of the momentum density or kinetic energy density in the continuum model.)

B. Two subtle points

First, we have seen from Fig. 5(b) that the L wave shifts masses to the left of their equilibrium positions and the succeeding T wave shifts the masses to the right. Referring back to our discussion at the end of Sec. VI, it is thus apparent that both the L wave and the T wave considered individually carry net longitudinal momenta. We have also seen that the longitudinal momentum carried by the T wave is related to the transverse motion of the string through the relation $G_T = -\int_{\mathrm{T\ wave}}(1/2)\rho_0\eta'\dot{\eta}\,dx$. What of the total momentum G_L carried by the L wave, though? At first sight, one might (in the case studied here) expect it to carry an equal but opposite momentum to the T wave, as the passage of both waves leaves each mass back at its starting position. However, because the L wave and T wave travel with different propagation velocities, the longitudinal motions in each are not (necessarily) mirror images of each other, and so the overall momentum need not be zero. [For the example given here, the overall longitudinal momentum is in fact negative, because $G_L = -(c_L/c_T)G_T$ for the way these waves were generated.) This is to be expected—the net longitudinal momentum carried by both waves must depend on the longitudinal forces and speeds used to generate the initial wave.

This is a subtle point: Note that moving the leftmost mass in our string vertically stretches the spring between it and its neighbouring mass, and so it exerts a pull on this mass, i.e., there is a longitudinal or negative x-directed force on the second mass in the chain. Thus the total longitudinal impulse given to the chain by the transverse motion of the leftmost mass in the chain is given by $-\int(\tau_x(t)-\tau_0)dt$, where τ_x is the x component of the tension in the leftmost spring in the chain. Evaluating this impulse numerically for the motion used to generate the waves shown in Fig. 5, we find that it agrees, as expected, with $\Sigma_j m_j \dot{x}_j$, the total longitudinal momentum carried by both the L wave and the T wave.

The second point that we wish to make is that the alert reader may also be wondering why the T-wave component of Fig. 5(d) is asymmetric. This asymmetrisation of the T wave is due to dispersion, which is of course not included in classic string theory. Since classic string theory is just the long wavelength limit of our model, however, this effect can of course be reduced by making the pulse broader, but only at the expense of longer computational times. (Notice also that because $c_L > c_T$, the precursor longitudinal wave is broader than the transverse wave, and asymmetrisation due to dispersion is not yet visible in it.)

C. Resolving paradox 2

We are now in a position to resolve paradox 2. First we note that it *is* true that a transverse wave *does* shift each element of string away from where it found it and in the direction of propagation because it is indeed accompanied by a longitudinal wave. However, because the generation of a transverse wave necessarily generates another longitudinal wave which separates off moving at speed c_L and which shifts every element of string it meets in the opposite direction to its propagation, the net effect of the passage of both waves is to leave each element of string back in its equilibrium position. Thus there is no differential tension set up in the plucked string example of Sec. IV B after both of the waves have passed by, as verified by further simulations.

D. Understanding the waves involved

In summary then, simulations of the type shown in Fig. 5 show that when we try to set up a transverse wave on an elastic string, we in fact generate two waves which travel independently of each other. The faster of these waves travels with the speed c_L and is a purely longitudinal mode of vibration. For convenience we shall refer to this wave as an ''L wave.'' The slower of the two waves generated travels at the speed c_T and includes both longitudinal as well as transverse motion of the particles. Since the dominant motion is transverse and the wave travels at the transverse wave speed c_T, we call these waves ''T waves.'' (Note that our L and T waves are a bit like the P and S waves of seismology, although since the T wave has both transverse and longitudinal components, it is perhaps more akin to the Rayleigh surface waves of seismology.[25,31] L and T waves as we describe have in fact been posited before in a theoretical analysis by Broer,[15] though the work of this author does not appear to be well known.)

Why is there an L wave associated with the T wave, though? Well, recall that stretching or ''compressing'' (actually relaxing the tension in) the string longitudinally will generate a longitudinal wave in the string. Consequently, since moving the left-hand end of the string vertically upwards necessarily stretches the string, doing so inescapably generates a longitudinal wave along with the transverse wave.

From Fig. 5(c), we see that our Gaussian T wave shifts each mass in the lattice to the right as well as up and down. One might presume therefore, that moving the left-hand end of the lattice to the right in just the correct way when moving it up and down, should produce a T wave only. This is in fact the case. (From the preceding paragraph, we can see that a pure T wave will be generated if the string is moved transversally and to the right so that the string is never stretched—see Appendix B 2.) What's the correct way to move the lattice to the right, though? Well, the simulations show that the longitudinal velocity profile of a T wave is given by $g/(2\rho_0)$, so we should move the lattice to the right with a velocity which matches this, i.e.,

$$\frac{\partial \xi}{\partial t} = \frac{1}{2c_T}\left(\frac{\partial \eta}{\partial t}\right)^2, \tag{23}$$

where $\partial\eta/\partial t$ is the time derivative of the transverse displacement given to the left-hand end of the lattice. [Note that because this formula has to be evaluated at the left-hand end of the lattice, the $\partial\eta/\partial x$ in the formula for g had to be transformed to $-(\partial\eta/\partial t)/c_T$, assuming $\eta = f(x - c_T t)$ is a valid solution. A theoretical discussion leading to Eq. (23) is given in Appendix B.]

We observe that the motion of the left-hand end of the string needed to produce a pure T wave in the form of a Gaussian pulse is shown by the T-wave part of Fig. 5(c). Note that since this wave shifts masses to the right of their equilibrium position, it carries a net momentum.

E. Resolving paradox 1

We can now address paradox 1: why wave momentum (which is the momentum carried by the T wave) is not conserved at a discontinuity in the mass density of the string and where the ''force'' f in Eq. (16) comes from. It is now probably obvious from the preceding discussion that (T-) wave momentum is not conserved at a discontinuity *because the element of string at the discontinuity cannot simultaneously move so as to produce a pure T-wave for both the reflected and transmitted waves*, and thus new L waves are generated at the discontinuity. The ''force density'' in Eq. (16) is thus seen to be (twice) the force the L waves generated at the discontinuity apply to the T waves generated at the discontinuity. This is verified by simulations.

F. Special case—the $c_T = c_L$ limit

From Appendix B, we see that $c_T = c_L$ when $a = 0$, or equivalently, when $SY = 0$. This condition is approximately met by the Slinky spring which has effectively a zero relaxed length.[32–34] In this case, the T wave and L wave travel together. Thus in the situation discussed above where a wave is excited by moving one end of the Slinky transversely, the L wave moves particles to the left and the T wave puts them back again simultaneously, thus resulting in a purely transverse motion of the elements of the Slinky spring.[32,33] Since the particles in the spring move purely transversely, *such a wave carries no longitudinal momentum*. This can be checked in the simulations by reflecting such a wave off a heavy particle placed at the opposite end of the Slinky. (Of course, a wave with longitudinal momentum *can* be generated in a Slinky by also moving the wave generating end longitudinally as well as transversely.)

Before we finish this special case, we'd like to look at an argument put forward by Pierce.[2] He states that it can be seen that a transverse wave on a string carries longitudinal momentum by the following argument. When a transverse pulse is reflected off a fixed end, the string makes an angle to the wall. The component of the string tension in the longitudinal direction is thus less than it was when the pulse was not present (assuming the tension in the string to be unchanged by the presence of the pulse). This reduction in tension is equivalent to a force against the wall and this force is due to the momentum carried by the wave.

The error in this argument is easily seen in the $c_T = c_L$ limit. Using the model described with Eqs. (1) in this case for a purely transverse wave, it is easily shown that there is an increase in the string tension due to the wave, and that this increase in tension exactly compensates for the angle that the string makes to the fixed end.[34] Thus in this case, there is no change in the longitudinal component of tension as the pulse gets reflected from the fixed end, and so the wave carries no longitudinal momentum as stated above. We conclude therefore, that while it *is* a valid approximation to neglect changes in string tension when deriving the transverse wave equation [Eq. (2)], these changes in tension are of crucial importance when it comes to the analysis of longitudinal momentum.

G. Seeing it all mathematically

With the results of the simulations to guide us, we can now see how the continuum wave equations can lead us to the same conclusions. In the continuum limit, the appropriate wave equations for the transverse $\eta(x,t)$ and longitudinal $\xi(x,t)$ displacements of the string are (see Appendix B):

$$\frac{\partial^2\eta}{\partial t^2} = c_T^2 \frac{\partial^2\eta}{\partial x^2}, \tag{24}$$

$$\frac{\partial^2\xi}{\partial t^2} = c_L^2 \frac{\partial^2\xi}{\partial x^2} + (c_L^2 - c_T^2)\frac{\partial\eta}{\partial x}\frac{\partial^2\eta}{\partial x^2}. \tag{25}$$

Equation (24) is the standard equation and we consider a travelling wave solution

$$\eta(x,t) = f(x - c_T t). \tag{26}$$

Equation (25) says that ξ will be a part $h(x - c_L t)$ propagating with the longitudinal speed c_L plus a part driven by the term involving η. This latter term we can write as $q(x - c_T t)$. Thus

$$\xi(x,t) = h(x - c_L t) + q(x - c_T t). \tag{27}$$

Substituting Eqs. (26) and (27) into (25) and putting $u = x - c_T t$ gives

$$\frac{d^2q}{du^2} = -\frac{df}{du}\frac{d^2f}{du^2}, \tag{28}$$

which integrates to

$$\frac{dq}{du} = -\frac{1}{2}\left(\frac{df}{du}\right)^2. \tag{29}$$

If we multiply by $\rho_0 c_T$, this equation is equivalent to

$$\rho_0\frac{\partial q}{\partial t} = -\frac{1}{2}\rho_0\frac{\partial f}{\partial t}\frac{\partial f}{\partial x} \equiv -\frac{1}{2}\rho_0\frac{\partial\eta}{\partial t}\frac{\partial\eta}{\partial x}. \tag{30}$$

Thus we see that the momentum carried by the longitudinal ξ wave driven or accompanied by the transverse η wave is exactly $(1/2)g$, as revealed by the simulations.

Thus our theory neatly confirms the general behavior deduced from the simulations—namely, that the motion of a string involves a transverse wave and two longitudinal waves, one travelling with speed c_L and the other travelling together with the transverse wave at speed c_T. The connection between the real momentum of the T wave in the x direction and the ''momentum'' derived for transverse waves is now very clear and Eq. (25) shows that there is a genuine link, not some mere coincidence, as the transverse waves may be seen to drive a longitudinal wave.

H. Relative magnitudes and importance

One last question remains though, and that is why aren't these results already well known? The answer has to do with the relative size and importance of effects. First the relative size.

The total transverse and longitudinal kinetic energies in the waves are given by $K_T = \int_{\mathrm{T\ wave}} (1/2)\rho_0 \dot{\eta}^2\, dx$

$\equiv\Sigma_j(1/2)m_j\dot{\eta}_j^2$ and $K_L=\int_{\mathrm{T\ wave+L\ wave}}(1/2)\rho_0\dot{\xi}^2\,dx \equiv\Sigma_j(1/2)m_j\dot{\xi}_j^2$, respectively. For the simulation shown in Fig. 5, $K_L/K_T\simeq 2.3\times 10^{-4}$, and thus the amount of energy carried by the longitudinal motions cogenerated with the transverse motion is negligibly small (except when magnified by the sound board of a musical instrument[28]) *provided all you are interested in is the transverse motion of the string*. That is to say, if one is only interested in the transverse wave motion of the string (and that is what most textbook examples are only interested in), then the usual formulae for the transverse wave equation and for the amplitudes of the transverse waves reflected and transmitted at a discontinuity in the string's mass density are all perfectly adequate. If, however, one wants to know anything about longitudinal motion of the string, such as the momentum carried by the string in the direction of propagation, *then a fully self-consistent theory must include all of the effects we have been talking about*.

Now the relative importance. Since the momentum carried by transverse waves is so small, and most of the questions of interest can be answered without recourse to momentum, wave momentum has not received a lot of attention and has hitherto been mostly of theoretical interest. And since the problem can't be solved exactly analytically, many previous theoretical treatments, *without the support of simulations*, have made errors in their underlying assumptions. Wave momentum may remain to be of theoretical interest for transverse waves (but it's good to get that theory right!!) though it is receiving increased attention for longitudinal waves in nonlinear applications.[7–9] It should also be noted that precursor longitudinal waves *are* of importance in creating the characteristic tones of the various stringed musical instruments because of the magnifying effects of the sound boards of these instruments.[28]

VIII. CONCLUDING DISCUSSION

We have discovered that with regard to the concept of the longitudinal momentum carried by a ''transverse'' wave on a taut string, the literature is confused and contradictory. It is widely claimed, e.g., Refs. 10, 12–14 that the wave momentum density is given by $g=-\rho_0(\partial\eta/\partial x)(\partial\eta/\partial t)\equiv-\rho_0\times\eta'\dot{\eta}$ or equivalently that the total momentum carried by the wave is given by $G=E/c_T$, where E is the total (kinetic K plus potential U) energy carried by the wave. This is also the wave momentum given by the canonical energy–momentum tensor when transverse motion alone is considered. Juenker,[3] on the other hand, claims that the answer depends on the properties of the string and on how the wave is initiated. He considers explicitly two examples: (i) waves on a rope-like (i.e., inextensible) string, for which $G=K/c_T$ and (ii) waves on a Slinky spring (for which $c_L=c_T$) for which the relationship between energy and wave velocity depends on the relative admixture of L and T waves (our terminology) in the wave. Broer[15] also posits the existence of what we call T waves, and shows that for these waves, $G=K/c_T$, noting their correspondence to waves on an inextensible string. *Implicit* in Broer's analysis is that for T waves, $g=-(1/2)\rho_0\eta'\dot{\eta}$.

We have attacked a resolution of the confusion on several fronts. First, we showed that assuming $g=-\rho_0\eta'\dot{\eta}$ and neglecting what we've called L waves led to paradoxes. We then used numerical simulations of an ideal string (perfectly flexible with linear elasticity) to resolve the paradoxes and to guide our theoretical analyses.

The conclusions of this work are as follows. One can't, except in the case when $c_L=c_T$, excite a purely transverse unidirectional wave on an elastic string; longitudinal waves are inescapably generated as well. However, for finite wave pulses (except in the case when $c_L=c_T$) a pure L wave will separate from what we call a T wave, a wave which propagates with unchanging shape at the standard transverse wave velocity c_T. The dominant motion associated with a T wave is transverse, but it also includes a small longitudinal component which provides the longitudinal momentum of the wave. In fact, T waves propagate exactly like the ''transverse'' wave on the *inextensible* string discussed by Juenker[3] and Broer.[15] The longitudinal momentum carried by T waves is found numerically to be $G=K_T/c_T$, with the momentum density $g=-(1/2)\rho_0\eta'\dot{\eta}$. This momentum was also shown to be a real momentum, not just a pseudomomentum as is often found with longitudinal waves. On the other hand, the total longitudinal momentum carried by the L wave depends on how the waves are generated and on the physical parameter a/l (or equivalently on SY and τ_0). We have also presented new theoretical arguments to support the results of the numerical simulations. These results confirm and extend the analyses of Broer[15] and Juenker.[3]

Further, we have discovered that it is possible to generate in the simulations a pure T wave at one end of a string. When this wave hits a discontinuity in the mass density of the string, L waves are generated in addition to the transmitted and reflected T waves. The energy of these L waves is small in comparison to the energy of the T waves, so to a first approximation, they may be ignored and the standard results for the amplitudes of the reflected and transmitted transverse waves hold. The L waves are *essential* however, for the total momentum of the system to be conserved, and their generation leads to the force density term in Eq. (16). L waves are also necessary to resolve the differential tension ''paradox'' of Sec. IV B.

Finally, what now is the status of the continuity equation (16) and the energy–momentum tensor given by Eq. (34)? Well, neither give the *actual* momentum density carried by T waves, though both, with a care for interpretation, can be used as valid calculational tools (see Secs. IV A and IV B).

It might be fitting to conclude with some words from The Master, Lord Rayleigh:

> [32, Vol. I, Chap. VI]. ''Among vibrating bodies there are none that occupy a more prominent position than Stretched Strings. From the earliest times they have been employed for musical purposes.... To the mathematician they must always possess a peculiar interest as a battle-field on which were fought out the controversies of D'Alembert, Euler, Bernoulli, and Lagrange relating to the nature of the solutions of partial differential equations. To the student of Acoustics they are doubly important.''

We trust that the smoke is now clearing from the battle over stretched-string wave momentum propagation.

APPENDIX A: LAGRANGIAN PERSPECTIVE

This Appendix follows Goldstein (Ref. 22, Chap. 12), and for notational convenience we introduce the following subscript notation: $x_0\equiv t$, $x_1\equiv x$, $\eta_{,\mu}\equiv\partial\eta/\partial x_\mu$; the greek subscripts μ, ν will take on the values 0 and 1; and the Einstein summation convention over repeated subscripts will be assumed.

Consider a general Lagrangian density with the functional dependence $\mathcal{L}=\mathcal{L}(\eta,\eta_{,\nu},x_\nu)$, $\nu=0,1$. Taking the total derivative $d\mathcal{L}/dx_\mu$ and using the Lagrange equations of motion, i.e.,

$$\frac{d}{dx_\nu}\left(\frac{\partial\mathcal{L}}{\partial\eta_{,\nu}}\right)-\frac{\partial\mathcal{L}}{\partial\eta}=0, \tag{31}$$

the following two 2-divergences for the energy–momentum tensor $T_{\mu\nu}$ for the field may be derived:

$$\frac{dT_{\mu\nu}}{dx_\nu}=-\frac{\partial\mathcal{L}}{\partial x_\mu}, \tag{32}$$

where

$$T_{\mu\nu}=\frac{\partial\mathcal{L}}{\partial\eta_{,\nu}}\,\eta_{,\mu}-\mathcal{L}\delta_{\mu\nu} \tag{33}$$

and $\delta_{\mu\nu}$ is the Kronecker delta symbol.

From Eq. (4), we can see that the Lagrangian density for transverse waves on a stretched string is just

$$\mathcal{L}=\tfrac{1}{2}\rho_0\,\eta_{,0}^2-\tfrac{1}{2}\tau_0\,\eta_{,1}^2. \tag{34}$$

Thus $T_{00}\equiv(\partial\mathcal{L}/\partial\eta_{,0})\,\eta_{,0}-\mathcal{L}=\epsilon$, the energy density of the wave; $T_{01}\equiv(\partial\mathcal{L}/\partial\eta_{,1})\,\eta_{,0}=-\tau_0\,\eta_{,1}\eta_{,0}$, the energy current density (power) of the wave; $T_{10}\equiv(\partial\mathcal{L}/\partial\eta_{,0})\,\eta_{,1}=\rho_0\,\eta_{,1}\eta_{,0}\equiv-g$, the negative of the momentum density of the wave; and $T_{11}\equiv(\partial\mathcal{L}/\partial\eta_{,1})\,\eta_{,1}-\mathcal{L}=-\epsilon$, the negative of the momentum current density of the wave.

The two-divergences, Eq. (32), thus lead in this case to (going back to our normal notation for clarity)

$$\frac{\partial\epsilon(x,t)}{\partial t}+\frac{\partial P(x,t)}{\partial x}=-\frac{\partial\mathcal{L}}{\partial t} \tag{35}$$

and

$$\frac{\partial g(x,t)}{\partial t}+\frac{\partial b(x,t)}{\partial x}=\frac{\partial\mathcal{L}}{\partial x}, \tag{36}$$

where we have introduced the symbol $b(x,t)$ for the momentum current density. (Note that for linear waves in a dispersionless medium, $b\equiv\epsilon$, but this result is not true in general.) These equations are basically Eqs. (14) and (16) from the main text, and we thus see that energy will be conserved if the Lagrangian density is explicitly independent of time t, and momentum will be conserved if the Lagrangian density is explicitly independent of position x. Thus from Eq. (34), we see that if $\rho=\rho(x)$, energy will be conserved but momentum will not, a point alluded to by some authors,[22,35] but not discussed in any detail. Note also that with $\rho=\rho(x)$ in Eq. (34), Eq. (36) reduces to Eq. (16) in the main text.

Finally, it should be emphasised that the Lagrangian in Eq. (34) is the one that leads to the standard transverse wave equation. A symmetry property—invariance under space translation—leads to a quantity which is labelled ''momentum.'' However, the link to the concept of momentum as used in dynamics is not explicitly made in this theory.

APPENDIX B: THREE DERIVATIONS OF THE CORRECT FORMULA FOR WAVE MOMENTUM

1. Via the coupled wave equations

A more accurate set of equations governing the propagation of waves on a taut string than Eq. (2) can be found by taking the continuum limit of Eqs. (1). In Eqs. (1), (x_j,y_j) are the spatial frame coordinates of the mass m_j, whose (assumed fixed) material frame coordinates are $(X_j,0)$. Letting (ξ_j,η_j) be the relative displacement coordinates of the mass m_j, then $x_j\equiv X_j+\xi_j$ and $y_j\equiv\eta_j$. Rewriting Eqs. (1) in terms of ξ_j and η_j and making second-order Taylor series approximations,

$$x_{j\pm1}\simeq X_j\pm l+\xi_j\pm\xi_j'l+\tfrac{1}{2}\,\xi_j''l^2, \tag{37a}$$

$$y_{j\pm1}\simeq\eta_j\pm\eta_j'l+\tfrac{1}{2}\,\eta_j''l^2, \tag{37b}$$

we find, to lowest order, that

$$\rho_0\ddot{\xi}\simeq(SY+\tau_0)\xi''+SY\eta'\,\eta'', \tag{38a}$$

$$\rho_0\ddot{\eta}\simeq\tau_0\,\eta''+SY(\tfrac{3}{2}\,\eta'^2\,\eta''+\xi'\,\eta''+\eta'\,\xi'')\simeq\tau_0\,\eta'', \tag{38b}$$

where $\rho_0\equiv m/l$, the equilibrium mass density of the string; $\tau_0\equiv k(l-a)$, the equilibrium tension in the string; and $SY=ka$ is the product of the cross-sectional area S and the Young's modulus Y of the string. (Note that these equations differ from those proposed in Refs. 10 and 14, but are believed to be the correct equations by virtue of their agreement with the numerical simulations. They are also consistent, when the approximations $|\eta'|\ll1$ and $|\xi'|\ll1$ are made, with the equations derived by Morse and Ingard[36] in a treatment of the nonlinear effects experienced by *large* amplitude waves propagating along ideal strings. We stress, though, that the importance of these equations with regard to the longitudinal momentum carried by small amplitude waves was not realised by these authors.)

Equation (38b) is of course, the standard transverse wave equation for waves propagating at the wave speed $c_T=\sqrt{\tau_0/\rho_0}$. When $\eta\to0$, Eq. (38a) reduces to the standard wave equation for longitudinal waves on a taut string[16] propagating with the wave speed $c_L=\sqrt{(SY+\tau_0)/\rho_0}$. Note that the tension increases c_L over its value for longitudinal waves in an unstressed solid. (Note also that Refs. 10 and 14 make the approximation $c_L\simeq\sqrt{SY/\rho_0}$, assuming that $\tau_0\ll SY$.)

As an aside, and tying up a loose end from the discussion in Appendix A, using Eqs. (37) in the model which gives Eq. (1), we find that the Lagrangian density for the string *including the effects of longitudinal stretching*, is given by

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}\,\rho_0(\dot{\xi}^2+\dot{\eta}^2)-\frac{1}{2}\,kl\left(\sqrt{(1+\xi')^2+\eta'^2}-\frac{a}{l}\right)^2\\&=\tfrac{1}{2}\,\rho_0(\dot{\xi}^2+\dot{\eta}^2)-\tfrac{1}{2}\,(SY+\tau_0)\\&\quad\times\left(\sqrt{(1+\xi')^2+\eta'^2}-\frac{SY}{SY+\tau_0}\right)^2,\end{aligned} \tag{39}$$

where the field variables ξ and η are considered to be functions of the independent variables time t and material coordinate X. Applying the Euler–Lagrange equations[22] to Eq. (39), and making our usual approximations $|\xi'|\ll1$ and

$|\eta'|\ll 1$, we again obtain Eqs. (38) to lowest order.

Our simulations suggest that when a transverse wave is present, then the total longitudinal displacement can be written as a sum of waves travelling with speeds c_L and c_T. Making this assumption, we substitute $\xi(x,t)=\xi_\mathrm{L}(x-c_\mathrm{L}t)+\xi_\mathrm{T}(x-c_\mathrm{T}t)$ into Eq. (38a), and, making use of Eq. (38b) and $\rho_0\ddot{\xi}_\mathrm{L}=(SY+\tau_0)\xi_\mathrm{L}''$, we find that $\xi_T''=-\eta'\eta''$, and so

$$\xi_\mathrm{T}'=-\tfrac{1}{2}\,\eta'^2. \tag{40}$$

For a wave of the assumed form, this result is equivalent to Eq. (23) in the main text, therefore Q.E.D.

Note also that in the limit $a\to 0$, $ka=SY\to 0$ and the longitudinal and transverse wave equations become completely independent of each other. Thus in this limit, a purely transverse wave becomes possible, and since such a wave has no longitudinal motion, it carries no longitudinal momentum.

2. Geometric argument

Consider a piece of string with its ends fixed a distance L_0 apart. After the generation of the wave (we presume caused by the lateral motion of the left-hand end) the length of the string will be

$$L=\int_0^{L_0}\sqrt{1+\eta'^2}\,dx\simeq L_0+\int_0^{L_0}\frac{1}{2}\,\eta'^2\,dx, \tag{41}$$

if $|\eta'|\ll 1$. As any stretching of the string will lead to a differential tension in the string and hence produce a longitudinal wave, we see that in setting up the wave, the left-hand end will have needed to move to the right by the amount $\Delta\xi=\Delta L\simeq\int_\mathrm{wave}(1/2)\eta'^2\,dx$ in order to maintain the total length of the string at L_0 and so avoid the production of a longitudinal wave. (Recall that this is in fact the procedure we described in Sec. VI D for generating a pure T wave—this is just another way of looking at that process.)

Now if $\eta=f(x-c_\mathrm{T}t)$, then $\eta'=-\dot{\eta}/c_\mathrm{T}$ and $\int_\mathrm{wave}(1/2)\eta'^2dx\equiv\int_\mathrm{wave}-(1/2)\eta'\dot{\eta}\,dt$, and since $\Delta\xi\equiv\int_\mathrm{wave}\dot{\xi}\,dt$, we thus have $\dot{\xi}=-(1/2)\eta'\dot{\eta}$ as required.

3. Inextensible rope argument

The wave set up by the argument in Sec. B 2 above, propagates in exactly the same fashion as a wave set up in an inextensible rope. Such waves have been analysed by Juenker,[3] and Broer,[15] who have shown that the x-propagating momentum of the wave is $G=K/c_\mathrm{T}$, where K is the total kinetic energy carried by the wave. Since the energy associated with the continuity equation for g is $K+U$, where U is the potential energy carried by the wave, and since $U=K$, it is perhaps not surprising then that g is twice the real physical momentum carried by the T wave.

[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison–Wesley, Reading, MA, 1964), Vol. II, Chap. 27.

[2] J. R. Pierce, *Almost All About Waves* (MIT, Cambridge, MA, 1974).

[3] D. W. Juenker, ''Energy and Momentum Transport in String Waves,'' Am. J. Phys. **44**, 94–99 (1976).

[4] Lord Rayleigh, ''On the Momentum and Pressure of Gaseous Vibrations, and on the Connexion with the Virial Theorem,'' Philos. Mag. **10**, 364–374 (1905).

[5] C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1986), 6th ed.

[6] I. H. Gilbert and B. R. Mollow, ''Momentum of Longitudinal Elastic Vibrations,'' Am. J. Phys. **36**, 822–825 (1968).

[7] V. L. Gurevich and A. Thellung, ''Quasimomentum in the theory of elasticity and its conservation,'' Phys. Rev. B **42**, 7345–7349 (1990).

[8] V. L. Gurevich and A. Thellung, ''On the quasimomentum of light and matter and its conservation,'' Physica A **188**, 654–674 (1992).

[9] D. F. Nelson, ''Momentum, pseudomomentum, and wave momentum: Toward resolving the Minkowski–Abraham controversy,'' Phys. Rev. A **44**, 3985–3996 (1991).

[10] W. C. Elmore and M. A. Heald, *Physics of Waves* (McGraw–Hill, New York, 1969).

[11] C. A. Coulson and A. Jeffrey, *Waves: A Mathematical Approach to the Common Types of Wave Motion* (Longman, London, 1977).

[12] A. Hirose and K. E. Lonngren, *Introduction to Wave Phenomena* (Wiley, New York, 1985).

[13] R. Benumof, ''Momentum propagation by travelling waves on a string,'' Am. J. Phys. **50**, 20–25 (1982).

[14] P. Stehle, ''The momentum of a transverse wave,'' Am. J. Phys. **55**, 613–615 (1987).

[15] L. J. F. Broer, ''On the Dynamics of Strings,'' J. Eng. Math. **4**, 195–202 (1970).

[16] G. R. Baldcock and T. Bridgeman, *Mathematical Theory of Wave Motion* (Ellis–Harwood, Chichester, 1981).

[17] S. Trillo and S. Wabnitz, ''Nonlinear Dynamics of Parametric Wave-Mixing Interactions in Optics: Instabilities and Chaos,'' in *Guided Wave Nonlinear Optics*, edited by D. B. Ostrowsky and R. Reinisch (Kluwer, Dordrecht, The Netherlands, 1992), pp. 489–534.

[18] C. Pask, D. R. Rowland, and W. Samir, ''A Constant of Motion for Modal Interactions in Nonlinear Dielectric Waveguides,'' J. Opt. Soc. Am. B **15**, 1871–1879 (1998).

[19] F. N. H. Robinson, ''Electromagnetic Stress and Momentum in Matter,'' Phys. Rep. **16**, 313–354 (1975).

[20] I. Brevik, ''Experiments in Phenomenological Electrodynamics and the Electromagnetic Energy-Momentum Tensor,'' Phys. Rep. **52**, 133–201 (1979).

[21] W. Shockley, ''A 'Try Simplest Cases' Resolution of the Abraham–Minkowski Controversy on Electromagnetic Momentum in Matter,'' Proc. Natl. Acad. Sci. USA **60**, 807–813 (1968).

[22] H. Goldstein, *Classical Mechanics* (Addison–Wesley, Reading, MA, 1980), 2nd ed.

[23] S. B. Palmer and M. S. Rogalski, *Advanced University Physics* (Gordon and Breach, New York, 1996).

[24] H. C. Corben and P. Stehle, *Classical Mechanics* (Wiley, New York, 1960).

[25] K. U. Ingard, *Fundamentals of Waves and Oscillations* (Cambridge U.P., Cambridge, 1988).

[26] S. Wolfram, *The Mathematica Book* (Cambridge U.P., Cambridge, 1996), 3rd ed.

[27] The $a/l=0$ limit is approximated by the Slinky spring, a popular waves demonstration ''toy,'' which has essentially a zero relaxed length and so for which $c_\mathrm{T}\simeq c_\mathrm{L}$. At the other extreme, a piano wire tuned to 440 Hz must be stretched by about 3% (Ref. 13).

[28] N. Giordano and A. J. Korty, ''Motion of a piano string: Longitudinal vibrations and the role of the bridge,'' J. Acoust. Soc. Am. **100**, 3899–3908 (1996).

[29] N. Giordano, ''The Physics of Vibrating Strings,'' Comput. Phys. **12**, 138–145 (1998).

[30] That the wave behaves like a particle with total momentum $G=\int_\mathrm{wave}\rho_0\dot{\xi}dx=-(1/2)\int_\mathrm{wave}\rho_0\eta'\dot{\eta}dx$, can be seen by simulating the reflection of a wave off a very large mass located at the right-hand end of the chain and attached to a fixed wall by a standard spring under the standard tension τ_0. This mass picks up momentum $2G$ as expected from the physics of particles. The mass needs to be very large so that in picking up this momentum, it hardly moves and so doesn't stretch the springs attached to it significantly.

[31] L. A. Segel, *Mathematics Applied to Continuum Mechanics* (Macmillan, New York, 1977).

[32] J. B. Keller, ''Large Amplitude Motion of a String,'' Am. J. Phys. **27**, 584–586 (1959).

[33] J. C. Luke, ''The motion of a stretched string with zero relaxed length in a gravitational field,'' Am. J. Phys. **60**, 529–532 (1992).

[34] E. A. Jagla and D. A. R. Dalvit, ''Null-length springs: Some curious properties,'' Am. J. Phys. **59**, 434–436 (1991).

[35] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[36] P. M. Morse and K. U. Ingard, *Theoretical Acoustics* (McGraw–Hill, New York, 1968).