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Small amplitude transverse waves on taut strings: exploring the significant effects of longitudinal motion on wave energy location and propagation

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Abstract

Introductory discussions of energy transport due to transverse waves on taut strings universally assume that the effects of longitudinal motion can be neglected, but this assumption is not even approximately valid unless the string is idealized to have a zero relaxed length, a requirement approximately met by the slinky spring. While making this additional idealization is probably the best approach to take when discussing waves on strings at the introductory level, for intermediate to advanced undergraduate classes in continuum mechanics and general wave phenomena where somewhat more realistic models of strings can be investigated, this paper makes the following contributions. First, various approaches to deriving the general energy continuity equation are critiqued and it is argued that the standard continuum mechanics approach to deriving such equations is the best because it leads to a conceptually clear, relatively simple derivation which provides a unique answer of greatest generality. In addition, a straightforward algorithm for calculating the transverse and longitudinal waves generated when a string is driven at one end is presented and used to investigate a $\cos^2$ transverse pulse. This example illustrates much important physics regarding energy transport in strings and allows the 'attack waves' observed when strings in musical instruments are struck or plucked to be approximately modelled and analysed algebraically. Regarding the ongoing debate as to whether the potential energy density in a string can be uniquely defined, it is shown by coupling an external energy source to a string that a suggested alternative formula for potential energy density requires an unphysical potential energy to be ascribed to the source for overall energy to be conserved and so cannot be considered to be physically valid.

1. Introduction

Elementary treatments of 'small' amplitude transverse waves on taut strings uniformly assume that longitudinal motion of points in the string can be neglected, such longitudinal motion typically only being considered in more advanced studies of 'large' amplitude waves. However, empirically, 'attack waves' travelling at the *longitudinal* wave speed have been found to contribute significantly to the characteristic tone of struck piano strings [1–3] and plucked guitar strings [4] and so these attack waves must carry non-negligible amounts of energy. In addition, the longitudinal momentum density of a transverse wave pulse calculated from an assumption of purely transverse motion predicts a dramatic failure of conservation of momentum when a pulse reflects off and transmits across a mass density change in a taut string [5]. These results indicate that longitudinal displacements of points in a string can have significant impacts on the observed physics, even when very small compared to the transverse displacements. Consequently, I argue that their neglect cannot be considered to be an idealization like neglecting friction or assuming the string is perfectly flexible (corrections to idealizations should lead to a 'fine tuning' of the predicted results, not to qualitatively quite distinct predictions).

On the basis of the above argument and since the study of waves in taut strings forms a fundamental part of the undergraduate physics curriculum, it seems important to get the basic physics correct and to be well aware of the conditions under which certain approximations are valid. However, while I have discussed the above issues in previous papers [5–7], the theoretical treatments have been rather complex mathematically in places and certain key results have only been demonstrated numerically by modelling a string as a chain of point masses joined by massless Hookean springs [5, 6]. For a theory which takes into account longitudinal motion to be *directly* useful in the classroom though, the exposition needs to be as mathematically simple and physically clear as possible. It would also be extremely useful if the coupled nonlinear wave equations have algebraic solutions that can be reasonably easily derived and analysed. A key goal of this paper then is to demonstrate that these things are possible.

In particular, section 2 shows that the standard continuum mechanics approach to deriving energy conservation equations for elastic continua provides a physically transparent and relatively mathematically simple derivation of the continuity equation for energy on a taut string taking into account fully three-dimensional motion of points in the string. Problematic aspects of alternative approaches are also discussed. In particular, it is argued that the Morse and Feshbach [8] alternative for the potential energy density formula can only be considered to be consistent with energy conservation for systems with non-rigid boundaries (such as are used to generate waves and the bridge–soundboard system of a musical instrument), even when integrated over the whole string, only if an unphysical potential energy is ascribed to the non-rigid boundary and hence cannot be considered to be physically valid.

Section 3 explores actual wave propagation when the linear transverse wave equation is approximately valid but transverse–longitudinal mode coupling is taken into account. An algorithm for finding solutions to the nonlinear coupled transverse–longitudinal wave equations when the string is driven at one end is derived and exact algebraic solutions for a $\cos^2$ transverse wave pulse coupled to a longitudinal wave are then obtained. These solutions provide algebraic support for the approximations made earlier in this paper and an algebraic demonstration of similar results obtained numerically by simulating a string as a chain of point masses joined by massless Hookean springs [5]. The simplifying approximation that the potential energy is a constant across the whole string is also briefly discussed, as is a possible approach to deriving the previously obtained standing wave solutions to the coupled transverse–longitudinal wave equations.

The key results of this paper are sufficiently intuitive that they can be discussed in a qualitative way at any level of instruction, though at the introductory level instructors might wish to avoid that complication. If that is the case, then in order to make the analyses given in introductory text books physically valid (e.g. [9–11]), the string being modelled needs to be idealized as having a zero relaxed length (see the discussion below (24)) in addition to being lossless and perfectly flexible. However, in intermediate to advanced undergraduate classes on general wave phenomena or continuum mechanics, the detailed analyses presented in this paper could find direct use, and to aid this some extra problems for students are suggested at various points in this paper.

2. Energy flow

2.1. Mathematically simple and physically transparent derivation of the exact continuity equation for energy

Derivations of the continuity equation for energy typically follow one of two approaches. One approach is to multiply the wave equation by the local string velocity and then use a differential identity to convert the resulting equation into an equation of the form of a continuity equation (e.g. section V of [12]). While such an approach can be generalized to take into account fully three-dimensional motion of elements of a string and not just purely transverse motion (section 5 of [7]), such derivations gloss over the fact that the differential identity one can use to obtain an equation of the right *general* form *is not unique* (see subsection 2.2), and so further physical reasoning is needed to determine which identity leads to the correct final equation.

A second approach to obtaining an energy continuity equation is to determine the potential energy density by a work argument; add this result to the kinetic energy density; partial differentiate with respect to time t the resultant total energy density and then use the wave equation and other results to rewrite the resulting expression as a spatial derivative (e.g. by dividing (2.1.2) in [8] by the length of the segment of string and taking the limit as the length of this segment goes to zero). Putting aside for the moment the ongoing debate as to whether the potential energy density can even be uniquely determined (this issue will be taken up in subsection 2.3), this approach suffers from the problem that the potential energy density for fully three-dimensional motion is somewhat complicated (see (8)) and the route to a continuity equation is complicated and unclear.

As a result of the above-mentioned weaknesses with alternative derivation approaches, the following standard result from the theory of elastic continua will be used: in the absence of body forces or thermal effects, the time rate of change of the total energy of a volume element of an elastic continuum is determined by the sum (integral) of the rates at which each surface force acting on the element does work on the element [13]. The exact result will be determined in order for the result to have maximum generality.

To start the analysis, consider the infinitesimal element of string with equilibrium length δX shown in figure 1. For a perfectly flexible string, the tension forces on each end of the element due to neighbouring elements act parallel or anti-parallel to the unit local tangent vector $\hat{\mathbf{s}}(X,t) = \mathbf{r}'/|\mathbf{r}'|$, where $\mathbf{r}(X,t) = [X + \xi(X,t),\ \eta(X,t),\ \zeta(X,t)]$ is the position at time t of the point in the string which has an equilibrium position of $[X,\ 0,\ 0]$ using a rectangular coordinate system with the x-axis[1] aligned along the equilibrium line of the string, and the prime indicates a partial derivative with respect to the material coordinate X. Let the magnitude of the exact tension in the string at the left hand of the segment be $\tau\ (X, t)$ (see appendix A) so that the force on the left-hand end of the element is given by $-\tau\hat{\mathbf{s}}$. On the right-hand end of the element the tension has a slightly different magnitude $\tau(X + \delta X, t) = \tau(X, t) + \delta\tau$ and a slightly different unit tangent vector $\hat{\mathbf{s}}(X + \delta X, t) = \hat{\mathbf{s}}(X, t) + \delta\hat{\mathbf{s}}$ (i.e. to keep the analysis

[1] The perspective taken in this paper is that X is the label of a point in the string. The x-coordinate of the point in the string with this label is given by $x(X, t) = X + \xi(X, t)$.

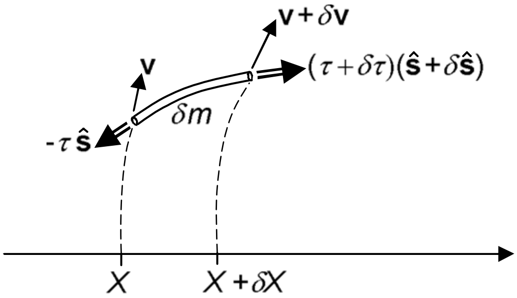

Figure 1. Motion of, and forces acting on, a small segment of string with mass δm which has an equilibrium position of X to $X + \delta X$ on the x-axis. Here $\mathbf{v}$ is the velocity of a point of the segment, τ is the local tension in the string and $\hat{\mathbf{s}}$ is the local unit tangent vector to the string. The forces (double arrows) acting on each end of the segment are elastic forces due to the neighbouring segments of the string.

'exact', no simplifying assumption that the tension remains approximately constant or that motion is purely transverse is being made). The force on the right-hand end of the element is thus $\tau\hat{\mathbf{s}} + \delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}} \approx \tau\hat{\mathbf{s}} + (\tau\hat{\mathbf{s}})'\delta X$ to first order in the infinitesimals.

Using the continuum mechanics result mentioned above [13], the time rate of change of the total energy, δE_T, of the element in the absence of body forces or thermal effects is given by the sum of the rates at which each surface force does work on the element. That is,

$$\partial(\delta E_T)/\partial t = -\tau\hat{\mathbf{s}}\cdot\mathbf{v} + (\tau\hat{\mathbf{s}} + \delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}})\cdot(\mathbf{v}+\delta\mathbf{v}) \approx (\delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}})\cdot\mathbf{v} + \tau\hat{\mathbf{s}}\cdot\delta\mathbf{v} \tag{1}$$

to first order in the infinitesimals. In (1), the fact that in general the ends of the element have slightly different velocities, $\mathbf{v}(X, t)$ and $\mathbf{v}(X + \delta X, t) = \mathbf{v}(X, t) + \delta\mathbf{v}$, respectively, has been taken into account.

To interpret (1) physically, note first that for free vibrations, figure 1 indicates that the net force on the segment is $\delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}}$ to first order, and hence by Newton's second law of motion, the acceleration of the element[2] is therefore given by

$$\delta m\dot{\mathbf{v}} = \delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}} \approx (\tau\hat{\mathbf{s}})'\delta X, \tag{2}$$

where $\delta m = \rho_0\delta X$ is the mass of the element, ρ_0 is the equilibrium mass per unit length of the stretched string, $\mathbf{v}(X, t) = \dot{\mathbf{r}}(X, t)$ is the velocity of the point of the string which has the equilibrium position of $[X, 0, 0]$ and the overdot represents a partial derivative with respect to time t. From (2), the first term on the right-hand side of (1) is consequently equal to

$$\delta m\dot{\mathbf{v}}\cdot\mathbf{v} = \frac{\partial}{\partial t}\left(\frac{1}{2}\delta m\mathbf{v}^2\right), \tag{3}$$

the time rate of change of the *kinetic* energy, δE_k, of the element (and thus the statement in [15] that the *net* force on a segment changes its *total* energy, i.e. its potential as well as its kinetic energy, is seen to be incorrect).

Since the first term on the right-hand side of (1) is just the time rate of change of the kinetic energy of the segment, the second term on the right-hand side of (1), $\tau\hat{\mathbf{s}}\cdot\delta\mathbf{v}$, should give the time rate of change of the potential energy of the segment. To see that this is so, note that $\delta\mathbf{v} = \partial(\delta\mathbf{r})/\partial t$, where $\delta\mathbf{r} = \mathbf{r}(X + \delta X, t) - \mathbf{r}(X, t)$ is the vector separation of the ends of the segment. Thus

$$\tau\hat{\mathbf{s}}\cdot\delta\mathbf{v} = \tau\hat{\mathbf{s}}\cdot\frac{\partial(\delta\mathbf{r})}{\partial t} \tag{4}$$

[2] Actually, strictly speaking the rate at which the net force does work on an elastic body gives the time rate of change of the centre of mass kinetic energy of the body [14], but for an infinitesimal element this distinction is not important.

is the rate at which the local tension force acting along the string does work in stretching the element, and thus it makes sense to interpret (4) as $\partial(\delta E_p)/\partial t$, the time rate of change of the potential energy, δE_p, of the element.

To obtain the continuity equation for energy, write $\delta \mathbf{v} \approx \mathbf{v}'\delta X$ and $\delta\tau\hat{\mathbf{s}} + \tau\delta\hat{\mathbf{s}} \approx (\tau\hat{\mathbf{s}})'\delta X$ in (1), divide by δX and take the limit as $\delta X \to 0$ to obtain

$$\frac{\partial \varepsilon_T}{\partial t} = (\tau\hat{\mathbf{s}})' \cdot \mathbf{v} + \tau\hat{\mathbf{s}} \cdot \mathbf{v}' = \frac{\partial}{\partial X}(\tau\hat{\mathbf{s}} \cdot \mathbf{v}), \tag{5}$$

where ε_T is the total energy density (i.e. energy per unit length) of the vibrating string. Equation (5) is the desired exact expression for the energy continuity equation for waves on a perfectly flexible, lossless taut string[3] and has the advantages over (18) in [7] of being in a conceptually clearer form and being valid even when the string is not linearly elastic.

Exploring (5) further, from the preceding discussion it follows that

$$\frac{\partial \varepsilon_p}{\partial t} = \tau\hat{\mathbf{s}} \cdot \frac{\partial \mathbf{v}}{\partial X} = \tau \frac{\mathbf{r}'}{|\mathbf{r}'|} \frac{\partial \mathbf{r}'}{\partial t} = \tau \frac{\partial |\mathbf{r}'|}{\partial t}, \tag{6}$$

which implies that

$$\tau = \frac{\partial \varepsilon_p}{\partial |\mathbf{r}'|}. \tag{7}$$

Equation (7) reflects the continuum mechanics result that the components of the stress tensor can be obtained from a stored energy function by differentiating with respect to the components of the strain tensor [17], with $|\mathbf{r}'|$ being related to the strain, $e(X, t)$, by $|\mathbf{r}'| = (1 + e)/(1 + e_0)$, where e_0 is the equilibrium strain defined above (8).

Using (A.1) for τ, it follows from (7) that for a linearly elastic, perfectly flexible string with Young's modulus Y, cross-sectional area S and tension $\tau_0 = SYe_0$ when no waves are present, the exact expression for the potential energy density in a wave is given by

$$\varepsilon_p(\text{wave}) = \tfrac{1}{2}(\tau_0 + SY)\mathbf{r}'^2 - SY|\mathbf{r}'| + \tfrac{1}{2}(SY - \tau_0), \tag{8}$$

where the integration constant follows from requiring $\varepsilon_p = 0$ when no waves are present (i.e. when $|\mathbf{r}'| = 1$). Thus for a linearly elastic, perfectly flexible string, ε_T in (5) is given by

$$\varepsilon_T = \tfrac{1}{2}\rho_0\dot{\mathbf{r}}^2 + \tfrac{1}{2}(\tau_0 + SY)\mathbf{r}'^2 - SY|\mathbf{r}'| + \tfrac{1}{2}(SY - \tau_0). \tag{9}$$

As will be explained below (24), purely transverse motion is only possible if $Y = 0$. When $Y = 0$, (9) reduces to the standard result, $\varepsilon_T = \frac{1}{2}\rho_0\dot{\eta}^2 + \frac{1}{2}\tau_0\eta'^2$, for purely transverse motion in one plane only.

2.2. The problem with trying to derive the energy continuity equation from the wave equation

Dividing (2) by δX and remembering that $\rho_0 = \delta m/\delta X$ is the equilibrium mass per unit length of the string, the exact wave equation for a lossless, perfectly flexibly string

$$\rho_0\dot{\mathbf{v}} = \partial(\tau\hat{\mathbf{s}})/\partial X \tag{10}$$

is obtained ([18] equation (14.3.4)). In elementary treatments though, purely transverse motion only is assumed. An energy continuity equation valid when transverse motion only is valid is then often obtained by multiplying the scalar wave equation by the transverse velocity of

[3] An equivalent form of (5) has been previously obtained using variational techniques and (7) (equation (23 in [16]), but this result does not appear to be well known. The approach presented above is preferable pedagogically though for at least two reasons. First, it does not require the more advanced mathematics of variational calculus, and second, the variational approach of [16] starts from the assumption that ε_p is a function of $|\mathbf{r}'|$, while this result is an outcome of the above analysis (see (7)).

the string and then making use of a differential identity. Following this process here, take the inner product of (10) with $\mathbf{v}$ to obtain

$$\rho_0 \dot{\mathbf{v}} \cdot \mathbf{v} = (\tau \hat{\mathbf{s}})' \cdot \mathbf{v}. \tag{11}$$

Writing the right-hand side of (11) as

$$(\tau \hat{\mathbf{s}})' \cdot \mathbf{v} = \frac{\partial}{\partial X}(\tau \hat{\mathbf{s}} \cdot \mathbf{v}) - \tau \hat{\mathbf{s}} \cdot \mathbf{v}', \tag{12}$$

the previously obtained energy continuity equation, (5), is obtained.

As the above approach seems much more efficient than the approach outlined in section 2.1, why is this approach not to be preferred? The problem is that in trying to obtain an expression in the form of (5), the right-hand side of (11) could equally well have been written as

$$(\tau \hat{\mathbf{s}})' \cdot \mathbf{v} = (\tau \hat{\mathbf{s}})' \cdot \dot{\mathbf{r}} = \partial[(\tau \hat{\mathbf{s}})' \cdot \mathbf{r}]/\partial t - \partial(\tau \hat{\mathbf{s}})'/\partial t \cdot \mathbf{r}, \tag{13}$$

and then using (A.1), one can, after a few more steps, determine that for a linearly elastic string

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0 \dot{\mathbf{r}}^2 - (\tau \hat{\mathbf{s}})' \cdot \mathbf{r} - \frac{1}{2}(SY + \tau_0)\mathbf{r}'^2\right] = -\frac{\partial}{\partial X}\left[\frac{\partial(\tau \hat{\mathbf{s}})}{\partial t} \cdot \mathbf{r}\right], \tag{14}$$

which suggests that $\varepsilon_p = -\left[(\tau \hat{\mathbf{s}})' \cdot \mathbf{r} + \frac{1}{2}(SY + \tau_0)\mathbf{r}'^2\right]$ rather than (8).

Furthermore, if, as is conventionally done, longitudinal motion is neglected and one starts with the linear transverse wave equation multiplied by the transverse velocity $\dot{\eta}$, that is

$$\rho_0 \ddot{\eta} \dot{\eta} = \tau_0 \eta'' \dot{\eta}, \tag{15}$$

where, as defined above, $\eta(X, t)$ is the transverse displacement, yet a third possibility can be obtained as follows. Split the right-hand side of (15) in half and on the first half use the approach indicated by (12) and on the second half use the approach indicated by (13). With one further step, the following 'energy' continuity equation is obtained:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 \dot{\eta}^2 - \frac{1}{2}\tau_0 \eta'' \eta\right) = \frac{\partial}{\partial X}\left(\frac{1}{2}\tau_0 \eta' \dot{\eta} - \frac{1}{2}\tau_0 \dot{\eta}' \eta\right). \tag{16}$$

Equation (16) is of particular interest because it suggests $\varepsilon_p = -\frac{1}{2}\tau_0 \eta'' \eta$, which Morse and Feshbach [8], and more recently Burko [19], have argued should be considered to be an acceptable alternative formula for potential energy density.

Clearly then, a unique energy continuity equation cannot be obtained by taking the inner product of the wave equation with the string velocity and using differential identities, and thus section 5 of [7] cannot be considered to provide independent support for a claim that (5) is the only physically valid energy continuity equation. The above analysis therefore indicates that either this general approach is not a reliable way to obtain the energy continuity equation, and hence the approach given in section 2.1 is to be preferred, or that Morse and Feshbach [8], Morse and Ingard ([18] p. 101) and Burko [19] were correct after all (apart from their neglect of longitudinal motion) that the potential energy density in string waves cannot be uniquely defined. New arguments *against* the latter possibility will be given in section 2.3.

Before leaving this section though, the following question might arise in readers' minds. Assuming that the potential energy density in a linearly elastic string is uniquely definable and that it is given by (8), one might wonder if the approach indicated by (12), of the three given above, does reliably give the right potential energy density. To see that it does not do so in all cases, consider purely longitudinal motion on a linearly elastic taut string for which the wave equation (10) simplifies to

$$\rho_0 \ddot{\xi} = (\tau_0 + SY)\xi'', \tag{17}$$

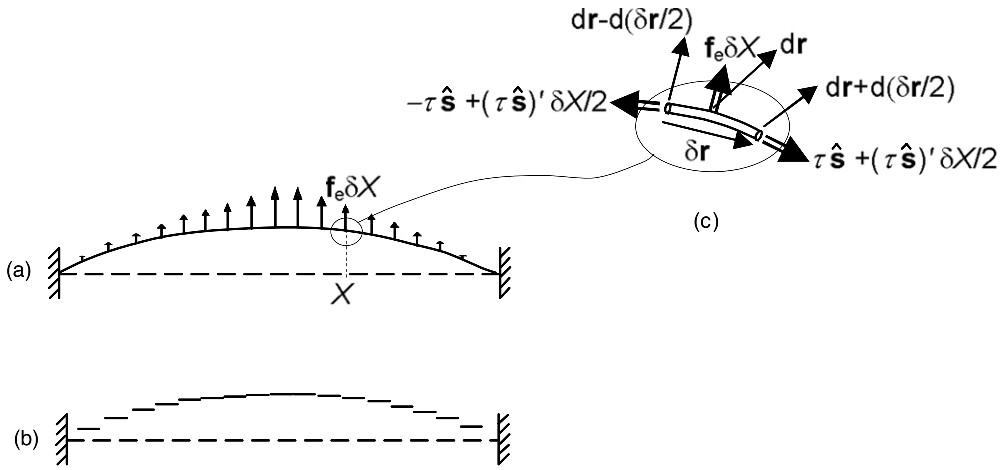

Figure 2. (a) Taut string fixed at each end held in quasistatic equilibrium by an external force density $\mathbf{f_e}\,(X)$ so that the total external force applied to an element of equilibrium width δX is $\mathbf{f_e}\,(X)\,\delta X$. (b) Implicit result of moving elements of a string without taking into account their changing orientation as they move. (c) Forces (double arrows) acting on, and infinitesimal displacements (single arrows) of, an element of string whose ends have a vector separation of $\delta\mathbf{r}$.

recalling that $\xi(X,t)$ represents the longitudinal displacement of a point in the string whose equilibrium x-coordinate is X. Multiplying (17) by $\dot{\xi}$ and then using the differential identity, $\xi''\dot{\xi} = (\xi'\dot{\xi})' - \xi'\dot{\xi}' = (\xi'\dot{\xi})' - \partial(\frac{1}{2}\xi'^2)/\partial t$, one would conclude, *erroneously*, that in this case $\varepsilon_p = \frac{1}{2}(\tau_0 + SY)\xi'^2$. However, from (B.3), the correct answer is $\varepsilon_p = \tau_0\xi' + \frac{1}{2}(\tau_0 + SY)\xi'^2$, with the difference arising because for a taut string undergoing purely longitudinal motion, (A.5) gives $\tau = \tau_0 + (\tau_0 + SY)\xi'$, not just $(\tau_0 + SY)\xi'$ as the right-hand side of (17) suggests.

Together with the arguments of section 2.3, the last result indicates that multiplying the wave equation by the string velocity and using differential identities is not a reliable method for obtaining the desired continuity equation and hence the approach of section 2.1 is to be preferred. Further support for this view comes from the fact that Benumof [20] used a similar approach to attempt to find an equation for the longitudinal momentum density associated with transverse waves from the scalar transverse wave equation, but the result was subsequently shown to fail to yield a conserved total momentum, despite what the equations suggested, and was in fact out by a factor of 2 and missing much important physics [5].

2.3. On why the potential energy density must be uniquely definable and the problem with nonstandard alternative formulas

An alternative approach believed to determine the potential energy density in a transverse wave on a string has been described by Morse and Feshbach [8]. This approach considers applying a distributed force across the string which varies in such a way that the string quasistatically moves from its original configuration to some final configuration, at all times being essentially in equilibrium (i.e. for each element of the string, the external force ‘exactly’ balances the net elastic force acting on that element—see figure 2(a)). Since the work done by this external force distribution does not change the kinetic energy of the string, it can be assumed to change the potential energy stored in the string.

Now, the work done by the part of this external force distribution acting on the element which has material coordinates $X - \delta X/2$ to $X + \delta X/2$ is given by $-\frac{1}{2}\tau_0 \eta''(X)\eta(X)\delta X$ when purely transverse motion only is assumed. The 'virtual displacement method' described by Mathews ([12], section III) also gives this potential energy density. However, since this result differs from the conventional transverse motion only result, $\frac{1}{2}\tau_0 \eta'^2 \delta X$, there has been a recent debate as to whether the Morse and Feshbach result can be considered to be correct and whether Morse and Feshbach were correct in claiming that the potential energy density cannot be uniquely defined [15, 19, 21] (and more generally whether the potential energy density in a string is only unique up to a term of the form $\partial\Phi/\partial X$, with Φ being a function which evaluates to zero for zero displacements ([12], p 101 of [18]). As agreement was not reached at the end of this debate [21], the purpose of this subsection is to add fresh insights to the discussion.

Now, while in principle the Morse and Feshbach approach could be redone taking into account fully three-dimensional motion of points in the string, the nonlinear nature of the full wave equation, (10), means that even if a closed-form algebraic expression for the potential energy density resulting from this approach exists, it would be horrendously complicated. A potential energy density *can* be obtained from the third-order approximate component wave equations (equations (23) and (24)), but the result is an unedifying mess. Consequently, for simplicity's sake, and also to be able to relate directly with previous results in the literature, in this subsection the string will be assumed to have a zero Young's modulus so that purely transverse motion of points in the string is possible (see comments under (24)).

The claim to be tested in this subsection is that the potential energy density in a wave cannot be uniquely specified. To test this claim, observe that for a unidirectional travelling wave, $\varepsilon_p = \frac{1}{2}\tau_0 \eta'^2$ gives a potential energy density which is always equal to the kinetic energy density (this follows from evaluating the densities for $\eta(X, t) = f(t \pm X/c_T)$, with $c_T = (\tau_0/\rho_0)^{1/2}$ being the transverse wave speed), and so this formula for ε_p predicts that the total energy in any segment of a pulse is *twice* that of the observable kinetic energy in that segment. In contrast, in the concave up parts of a bell-shaped pulse, $-\frac{1}{2}\tau_0 \eta''\eta$ is *negative*, and so this formula for ε_p predicts that the total energy in a concave up segment of a pulse is *less than* the observable amount of kinetic energy in that part of the pulse, leading to a greater than a *factor of two difference* in the predicted amounts of energy in the concave up parts of a pulse. One would think that this difference would have measurable consequences, such as in the power needed to create a travelling wave pulse.

To test this idea, consider the idealized device shown in figure 3 for generating a transverse wave pulse on a semi-infinite taut string with linear mass density ρ_0 and equilibrium tension τ_0. In this device, the system which provides the energy for the wave pulse consists of a mass m, which can slide without friction up and down a vertical pole, to which the left-hand end of the string is attached. The mass is also attached to a vertically supported spring with spring constant k which for the situation shown is initially stretched by an amount y_0 below the gravitational equilibrium of the spring–mass system. This initial stretch is arranged to be such that the string is initially at rest along the horizontal x-axis[4].

If the mass is released from rest at time $t = 0$, then the spring will initially cause the mass to accelerate upwards which will in turn cause a wave to start travelling down the string.

[4] A similar model which consists of a mass suspended by a spring and then connected to the centre of the string by another spring has been considered previously in [22] and [23]. Such a model avoids the contrivance of a frictionless pole, but leads to an equation of motion for the mass which is third order rather than second order in time (equation (32) in [23]). While this complication can be removed by replacing the spring between the mass and the string by a rigid couple, the model is still somewhat more complicated than the one considered here because waves travelling to both the left and the right need to be considered.

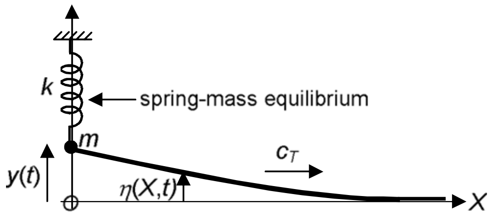

Figure 3. A mass m, constrained to slide vertically up and down a frictionless pole, is attached to a horizontal taut string and a vertically supported spring. For the situation shown, the mass was released from rest a distance y_0 below the spring–mass equilibrium position so that the mass is initially accelerated upwards and at the instant shown is still moving vertically upwards. The distance y_0 was chosen so that the string was initially at rest along the x-axis and thus the wave profile $\eta\,(X, t) = y\,(t - X/c_T)$, where $y(t)$ is the vertical height of m at time t.

Assuming small amplitude motions so that the linear transverse wave equation is valid, then the generated wave is given by

$$\eta(X, t) = y(t - X/c_T), \tag{18}$$

where $y(t)$ is the vertical height of the mass at time t (i.e. $y = 0$ has been defined to be $y(0)$). If the slope of this wave at the origin is small, then the vertical component of the tension force provided by the string on the mass can be approximated by $\tau_0\eta'(0, t) = -\tau_0\dot{y}/c_T = -\rho_0 c_T\dot{y}$, where the last expression was obtained by using $c_T^2 = \tau_0/\rho_0$. The equation of motion of the mass is thus

$$m\ddot{y} = -k(y - y_0) - \rho_0 c_T\dot{y}. \tag{19}$$

To determine the power the spring–mass system provides to create the travelling wave, multiply (19) by $\dot{y}$ to obtain

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{y}^2 + \frac{1}{2}k(y - y_0)^2\right] = -\rho_0 c_T\dot{y}^2. \tag{20}$$

By energy conservation, the physical interpretation of this equation is that the rate of change of the kinetic energy of the mass plus the potential energy in the spring equals the negative of the rate of change of the energy in the wave. Thus from (20), the rate at which the wave gains energy is given by$\rho_0 c_T\dot{y}^2$. Now for this particular problem, the rate of change of the energy in the wave is also given by the total wave energy density at $X = 0$ times the wave speed c_T. If the total wave energy density is taken to be $\varepsilon_T(X, t) = \frac{1}{2}\rho_0\dot{\eta}^2 + \frac{1}{2}\tau_0\eta'^2$, then by (18), $\varepsilon_T(0, t) = \rho_0\dot{y}^2$ and the right-hand side of (20) does indeed equal $-\varepsilon_T(0, t)c_T$. This is *not* the case though if the total wave energy density is taken to be $\varepsilon_T(X, t) = \frac{1}{2}\rho_0\dot{\eta}^2 - \frac{1}{2}\tau_0\eta''\eta$.

Does this result end the debate in favour of $\varepsilon_p = \frac{1}{2}\tau_0\eta'^2$ over $-\frac{1}{2}\tau_0\eta''\eta$ then? Not yet, for the right-hand side of (20) can be split in half with one of the halves rewritten using the differential identity, $\dot{y}^2 = d(y\dot{y})/dt - y\ddot{y}$, to obtain

$$\frac{d}{dt}\left[\frac{1}{2}m\dot{y}^2 + \frac{1}{2}k(y - y_0)^2 + \frac{1}{2}\rho_0 c_T y\dot{y}\right] = -\left(\frac{1}{2}\rho_0\dot{y}^2 - \frac{1}{2}\rho_0 y\ddot{y}\right)c_T, \tag{21}$$

and using (18), the right-hand side of (21) *is* $-\varepsilon_T(0, t)c_T$ if the total wave energy density is taken to be $\varepsilon_T(X, t) = \frac{1}{2}\rho_0\dot{\eta}^2 - \frac{1}{2}\tau_0\eta''\eta$. Does this then prove that $\varepsilon_p = -\frac{1}{2}\tau_0\eta''\eta$ is just as valid to use as a potential energy density as $\varepsilon_p = \frac{1}{2}\tau_0\eta'^2$? Before this conclusion can be drawn though, the $\frac{1}{2}\rho_0 c_T y\dot{y}$ term on the left-hand side of (21) needs to be given a physical interpretation; otherwise (21) is just a *mathematical* identity, not a *physical* equation.

Now the only possible *physical* interpretation of the $\frac{1}{2}\rho_0 c_T y\dot{y}$ term is that it is a potential energy which the mass gains as a result of being attached to the stretched string. This interpretation is untenable though for a number of reasons. First, as (19) shows, the force of the string on the mass is a purely dissipative force, and potential energies cannot be associated with dissipative forces. This can also be seen by trying to construct a Lagrangian for the mass using $\frac{1}{2}\rho_0 c_T y\dot{y}$ as a potential energy. However, the Lagrangian $L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}k(y - y_0)^2 - \frac{1}{2}\rho_0 c_T y\dot{y}$, when substituted into the Euler–Lagrange equation just gives $m\ddot{y} = -k(y - y_0)$ rather than (19). Thus $\frac{1}{2}\rho_0 c_T y\dot{y}$ *cannot* be interpreted as a potential energy for the mass, (21) is just a mathematical identity devoid of physical meaning and the rate at which the spring–mass system loses energy in creating a travelling wave is only consistent with $\varepsilon_p = \frac{1}{2}\tau_0\eta'^2$.

Note that these conclusions are not particular to the case chosen for investigation but are completely general. Different initial conditions for the spring–mass–string system lead to the same conclusions and it is also possible to analyse an arbitrary travelling wave reflecting off the spring–mass system which acts as an absorber[5]. In all cases, $\varepsilon_p = \frac{1}{2}\tau_0\eta'^2$ is consistent with energy conservation while the use of $\varepsilon_p = -\frac{1}{2}\tau_0\eta''\eta$ requires the introduction of an unphysical potential energy for the mass for total system energy to be conserved. Even for the case of a standing wave as Morse and Feshbach considered, if a boundary is allowed to be non-rigid, then $\eta\eta'$ does not necessarily equal zero at the non-rigid boundary and the two formulas for ε_p can only lead to the same total energy for the string plus boundary system if the boundary is allowed to have the unphysical potential energy discussed above. Thus, even if Morse and Feshbach's claim that only the energy in the string plus boundary has a unique value was correct, this claim still requires a unique string potential energy density.

To understand where the Morse and Feshbach [8] and virtual displacement [12] approaches have gone wrong, note that neither approach directly takes into account the fact that as an element of string moves, in general its orientation and hence length changes. Both approaches only indirectly take this change in orientation into account by virtue of the fact that neighbouring elements in general move slightly different amounts and so strictly speaking the final string configuration these approaches end up with is shown in figure 2(b) with the length of the elements made infinitesimal. Consequently, neither approach calculates directly the potential energy of stretching of an element. Note also from figure 2(c) that $\mathbf{f}_e$ is not the only force external to an element of string which does work on it as it moves. The interested reader can use the forces and displacements shown in figure 2(c), together with $\mathbf{f}_e = -(\tau\hat{\mathbf{s}})'\delta X$, to show that for an infinitesimal, quasistatic displacement, the change in potential energy of the element is given by $\tau\hat{\mathbf{s}} \cdot d(\delta\mathbf{r})$, the work done by the local tension force in stretching the element by an amount $d(\delta\mathbf{r})$, consistent with the theory presented in section 2.1.

To understand what these alternative approaches are measuring, note that there are two senses in which an element of string can have a potential energy. First, it can be considered to have potential energy by virtue of the fact that having been stretched as it moves, it is then capable of doing work on neighbouring elements as the amount of stretching is relaxed. Second, an element of string can be considered to have gained potential energy by virtue of it being attached to a stretched string which can do work on the element as the element moves. That is, the second approach treats the element as though it were an inextensible nugget

[5] In this case, one would have the gravitational equilibrium position of the spring–mass system at y = 0 and this problem would make an interesting student exercise. Another interesting exercise along these lines would be to model the bridge and sound board coupled to a string as the spring–mass system coupled to another semi-infinite string attached to the mass but heading in the negative x direction.

of matter attached to a spring (the rest of the string) with the nugget of matter rather than the spring being considered to have gained a potential energy $\frac{1}{2}kx^2$ by virtue of the spring stretching[6]. Thus the approaches can be said to indicate the potential energy *in* the element and the potential energy *of* the element. But since for calculations one can ascribe the potential energy of a mass–spring system to the mass or to the spring, why isn't it valid to do it either way for the string?

Feynman *et al* [24] addressed the equivalent question in the context of electrostatics where the question of whether the electrostatic (potential) energy of the system is to be found in the charges in the system or in the electric field between the charges arises. Of course, one can do *calculations* from either perspective[7], but is one perspective the more *physically* correct one? Feynman's conclusion was that since Einstein's equation of general relativity requires the location of non-gravitational energy to be uniquely determined, then *in principle* the energy content of a system should be uniquely localizable; otherwise different conventions for determining the location of potential energy in a system would (theoretically) yield gravitational fields with different physical effects. Feynman then goes on to consider the flow of energy in radiation to conclude that the electrostatic energy should be considered to be stored in the fields rather than the charges, as only that interpretation properly allows for the flow of energy in dynamic situations. In the case of waves on strings, this is akin to saying that the potential energy of an element is the energy stored in the element as a result of stretching rather than the energy the element gains by virtue of being attached to a stretched string. (From the perspective given above, it seems strange to say that the potential energy 'here' is not due to the stretching 'here', but due to stretching 'elsewhere' in the rest of the string. A striking example of this is to pluck a string in its middle with a narrow pick so as to create a triangular displacement. For this case, the $\varepsilon_p = \frac{1}{2}\tau_0\eta'^2$ formula indicates that the potential energy is uniformly distributed across the string, as one would expect based on a consideration of the electromagnetic bonds between neighbouring molecules in the string, while the $\varepsilon_p = -\frac{1}{2}\tau_0\eta''\eta$ formula suggests that the potential energy is zero everywhere except at the apex of the triangle.)

Finally, tying up a loose end from the debate, Morse and Feshbach [8] state that the two formulas for potential energy density differ because the energy of the 'ends' of a segment of string 'cannot be uniquely determined'. However, from a continuum mechanics perspective this statement makes no sense as in continuum mechanics, only volumes can have energy, not the surfaces of volumes, and the 'ends' Morse and Feshbach refer to are the bounding surfaces of a segment of string. A bounding surface is a mathematical surface with zero width and hence contains zero matter and hence can have no kinetic or potential energy associated with it. (In continuum mechanics, a 'point' which can have energy is an infinitesimal volume element, not an infinitesimal surface element.) Consequently, the expression $-\frac{1}{2}\eta'\eta]_{X=a}^{X=b}$ does not represent the value of anything at $X = \mathrm{a}$ and $X = \mathrm{b}$; it is merely a consequence of the mathematical fact that if two formulas for energy per unit length are to agree for arbitrary string displacement over the entire length of a string with fixed ends, then they can only differ by a function of the form $\partial\Phi/\partial X$, with Φ being an expression which evaluates to zero for zero displacements.

[6] This analogy is even more exact than this. If the mass attached to a spring with spring constant k stretched from equilibrium by a distance x is considered to have the potential energy $\frac{1}{2}kx^2$, then because the restoring force acting on the mass is $F = -kx$, the potential energy of the mass can be rewritten as $-\frac{1}{2}Fx$, i.e. the potential energy is negative one half of the force acting on the element times its displacement from equilibrium, and $-\frac{1}{2}\tau_0\eta''\eta\delta X$ has exactly the same interpretation.

[7] In the case of the string, the 'energy' continuity equation for the Morse and Feshbach result is (16).

3. Wave propagation

3.1. Introduction

Although the analyses in section 2 establish some important theoretical results and clarify some aspects of the physics of energy propagation in waves on strings, in practice energy propagation in waves is determined by first calculating the waves predicted by the wave equation with the appropriate boundary and initial conditions and any driving forces. The exact wave equation for a lossless, perfectly flexible string given by (10) is not in a form suitable for making such calculations though. To obtain equations which are, (10) is written in component form and approximations are made by assuming that the magnitudes of the spatial derivatives of the component displacements from equilibrium are everywhere small in comparison to unity. It has been shown previously that if the magnitudes of spatial derivatives satisfy [7]

$$\left(\frac{c_L^2 - c_T^2}{c_T^2}\right)|\xi'| \ll 1, \qquad \left(\frac{c_L^2 - c_T^2}{c_T^2}\right)\eta'^2 \ll 1 \quad \text{and} \quad \left(\frac{c_L^2 - c_T^2}{c_T^2}\right)\zeta'^2 \ll 1, \tag{22}$$

where $c_L^2 = (\tau_0 + SY)/\rho_0$ is the longitudinal wave speed in a stretched string, then transverse waves approximately obey the linear transverse wave equation

$$\rho_0\ddot{\eta} \approx \tau_0\eta'', \tag{23}$$

with an equivalent equation holding for $\zeta(X, t)$, and the longitudinal wave equation is given approximately by

$$\rho_0\ddot{\xi} \approx (\tau_0 + SY)\xi'' + SY(\eta'\eta'' + \zeta'\zeta''). \tag{24}$$

Sections 3.2 and 3.4 show how these nonlinearly coupled equations can be solved algebraically for a pulse and a standing wave with transverse vibrations in one plane only, while section 3.3 looks at their solution for a pulse when the uniform tension approximation is made. Note from (24) that energy is necessarily coupled from the transverse mode into the longitudinal mode unless $Y = 0$, a condition approximately met by the slinky spring [25], and thus strictly speaking, waves with purely transverse motion are only approximately possible for $Y \approx 0$ 'strings' like the slinky spring. A key result of this section will be that significant coupling of energy from the transverse modes of vibration to the longitudinal mode is *not* a 'large amplitude' effect, but will necessarily occur at all amplitudes.

3.2. Travelling wave pulse on a semi-infinite string

The analysis of travelling wave pulses is of considerable pedagogical interest because struck piano strings and plucked guitar strings produce pulses and so an analysis of pulses can be expected to give some physical insights into the waves found in musical acoustics. Furthermore, if a pulse on a semi-infinite string is considered, then because the longitudinal wave speed is greater than the transverse wave speed (except when $Y = 0$), it is possible for the longitudinal wave created by the transverse wave to completely separate from the transverse wave, thus allowing an unambiguous determination of the amount of energy that is coupled from the transverse mode to the longitudinal mode to be made. This determination is important because it points to why the longitudinal 'attack waves' mentioned in section 1 can have a significant impact on the sounds produced by stringed instruments. The example also illustrates how it is possible that transverse waves can be governed by the linear transverse wave equation *even though significant amounts of energy get coupled into longitudinal modes.*

Previously I have numerically simulated Gaussian transverse pulses on a taut string [5] but did not realize that algebraic solutions of the coupled wave equations were also possible, such algebraic solutions being of interest because they make explicit proofs of various results possible. How to obtain such algebraic solutions will be discussed in this subsection for the case of a semi-infinite string driven at one end. A string driven at one end rather than one plucked or struck will be considered so that only waves travelling in one direction will need to be considered. It also appears that while a straightforward algorithm for solving the coupled wave equations exists if the string is driven at a single point, this does not appear to be the case if the string is forced over a finite region. To be able to compare the obtained algebraic results with those obtained numerically for the Gaussian pulse, a $\cos^2$ transverse pulse will be considered since it has a similar bell-shaped profile to a Gaussian pulse but has the advantage that the required integrals have solutions in terms of elementary functions. Other types of pulses that can be investigated will also be mentioned.

The key to solving (23) and (24) for a finite unidirectional travelling wave is to recognize that physically, if the tension is not exactly uniform across the transverse wave, longitudinal waves will be created which will separate from the transverse wave until the tension across the transverse wave *is* exactly uniform and equal to the 'ambient' tension, τ_0. From (A.4), it follows that when this has occurred (to the level of approximation being considered), the longitudinal motion associated with a stable transverse wave will satisfy

$$\xi_T' = -\tfrac{1}{2}\eta'^2, \tag{25}$$

where the subscript T on ξ indicates that it refers to only that part of the longitudinal motion that travels at the transverse wave speed c_T (here, a pulse with displacements in a single transverse plane only will be considered for simplicity). Since unidirectional travelling wave solutions to (23) have the functional form $\eta = \eta(X - c_T t)$, it follows from (25) that $\xi_T = \xi_T(X - c_T t)$ as well. Once a longitudinal wave travelling at the longitudinal wave speed c_L has separated from the transverse wave, by (24) it satisfies $\rho_0\ddot{\xi} \approx (\tau_0 + SY)\xi''$, and hence this wave must have the functional form, $\xi_L = \xi_L(X - c_L t)$. The preceding analysis suggests that for a given $\eta = \eta(X - c_T t)$, the general solution to (24) is given by

$$\xi(X, t) = \xi_L(X - c_L t) + \xi_T(X - c_T t), \tag{26}$$

with ξ_T satisfying (25) and ξ_L satisfying $\rho_0\ddot{\xi} \approx (\tau_0 + SY)\xi''$ [5]. Substituting (26) into (24) verifies this assumption.

One last result is needed to determine an algorithm for solving (23) and (24) for a string driven at the end $X = 0$. This result is that since $\eta = \eta(X - c_T t)$, $\eta' = -\dot{\eta}/c_T$, and so at the boundary $X = 0$, $\eta'(0, t) = -\dot{\eta}(0, t)/c_T$. Similarly, $\xi_T'(0, t) = -\dot{\xi}_T(0, t)/c_T$, and so at the boundary $X = 0$, condition (25) becomes

$$\dot{\xi}_T(0, t) = \frac{\dot{\eta}^2(0, t)}{2c_T}. \tag{27}$$

The algorithm for solving (23) and (24) for a string driven at $X = 0$ is therefore as follows. If the waves in the string are fully caused by the boundary conditions $\eta(0, t)$ and $\xi(0, t)$, then

1. $\eta(X, t) = \eta(0, t - X/c_T)$
2. $\xi_T(0, t)$ can be determined by solving (27) and using $\xi_T(0, t_0) = 0$, where t_0 is the time when the boundary first starts to be displaced
3. $\xi_T(X, t) = \xi_T(0, t - X/c_T)$
4. $\xi_L(0, t) = \xi(0, t) - \xi_T(0, t)$
5. $\xi_L(X, t) = \xi_L(0, t - X/c_L)$.

The disturbances $\eta(X, t) = \eta(0, t - X/c_T)$ and $\xi_T(X, t) = \xi_T(0, t - X/c_T)$ travel together at the transverse wave speed c_T and comprise what will be referred to as a *T*-wave. The disturbance $\xi_L(X, t) = \xi_L(0, t - X/c_L)$ travelling at the longitudinal wave speed will be referred to as an *L*-wave.

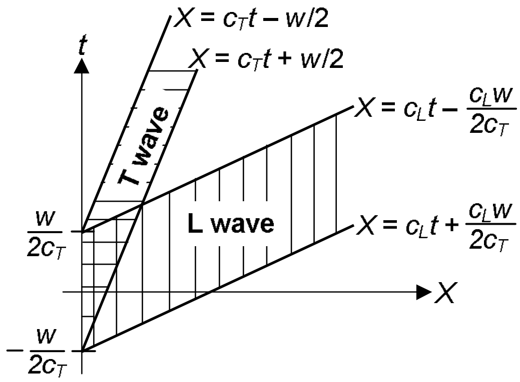

Figure 4. The cross-hatched areas are the regions in the tX-plane for which the T- and L-waves described by (30), (33) and (35) are non-zero.

To see how the above approach works in practice, and to determine how much energy gets coupled into an L-wave, consider a $\cos^2$ transverse travelling wave pulse excited by the $X = 0$ boundary conditions:

$$\eta(0,t) = \begin{cases} A\cos^2[\pi c_T t/w], & -\dfrac{w}{2c_T} \leqslant t \leqslant \dfrac{w}{2c_T} \\ 0, & \text{otherwise} \end{cases} \tag{28}$$

and

$$\xi(0,t) = 0. \tag{29}$$

By step 1 of the solution algorithm,

$$\eta(X,t) = \begin{cases} A\cos^2[\pi(X - c_T t)/w], & \text{in the } T\text{-wave region of figure 4} \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

as shown in figure 5(a).

By step 2,

$$\dot{\xi}_T(0,t) = \frac{\pi^2 c_T A^2}{2w^2}\sin^2(2\pi c_T t/w). \tag{31}$$

By using the trigonometric identity, $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, (31) can be integrated to obtain

$$\xi_T(0,t) = \frac{\pi^2 c_T A^2}{4w^2}\left[t - \frac{w}{4\pi c_T}\sin^2(4\pi c_T t/w)\right] + \alpha. \tag{32}$$

Since $\xi_T(0, -w/(2c_T)) = 0$, it follows that the integration constant $\alpha = \pi^2 A^2/(8w)$, and hence by step 3,

$$\xi_T(X,t) = \frac{\pi^2 A^2}{8w} - \frac{\pi^2 A^2}{4w^2}(X - c_T t) + \frac{\pi A^2}{16w}\sin\left[\frac{4\pi}{w}(X - c_T t)\right], \tag{33}$$

in the T-wave region of figure 4, or zero otherwise.

Since $\xi(0,t) = 0$, it then follows from step 4 that $\xi_L(0,t) = -\xi_T(0,t)$. Thus from (33),

$$\xi_L(0,t) = \begin{cases} -\dfrac{\pi^2 A^2}{8w}\left[1 + \dfrac{2c_T t}{w} - \dfrac{1}{2\pi}\sin\left(\dfrac{4\pi}{w}c_T t\right)\right], & -\dfrac{w}{2c_T} \leqslant t \leqslant \dfrac{w}{2c_T} \\ 0, \quad \text{otherwise.} \end{cases} \tag{34}$$

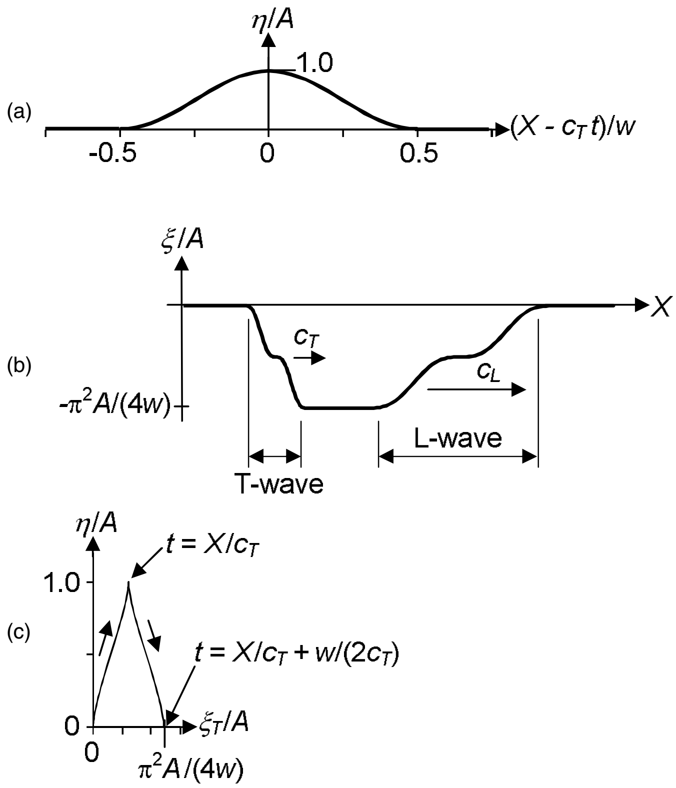

Figure 5. (a) Profile of the transverse wave pulse described by (30) for $t > w/(2c_T)$. Note that the horizontal and vertical axes are not to the same scale as the validity of (23) on which the analysis of this subsection is based requires that the pulse's amplitude, A, needs to be much smaller than its width, w. (b) Profile of the longitudinal displacements given by (33) and (35) at a time after the *T*- and *L*-waves have separated. Observe that the width of the leading pulse is c_L/c_T times that of the transverse pulse and that the longitudinal displacement is not zero at the trailing edge of the pulse. (For this example, $c_L/c_T = 3$ was chosen.) This result means that the *L*-wave corresponds to a segment of string which has been *stretched* relative to its equilibrium configuration. (c) Motion of a point in the string due to the passage of the *T*-wave given by (30) and (33) in combination. Note that the longitudinal displacement is of order A/w times smaller than the transverse displacement and is just what is necessary to reverse the negative longitudinal displacement caused by the passage of the *L*-wave shown in (b).

Finally, replacing t by $t - X/c_L$ in (34), one finds

$$\xi_L(X,t) = \begin{cases} 0, X > c_L[t + w/(2c_T)] \\ -\dfrac{\pi^2 A^2}{8w}\left\{1 - \dfrac{2c_T}{c_L w}(X - c_L t) + \dfrac{1}{2\pi}\sin\left[\dfrac{4\pi}{w}\dfrac{c_T}{c_L}(X - c_L t)\right]\right\}, \\ \qquad \text{in the } L\text{-wave region of figure 4} \\ -\dfrac{\pi^2 A^2}{4w}, \quad 0 \leqslant X \leqslant c_L[t - w/(2c_T)] \quad \text{when } c_L[t - w/(2c_T)] > 0 \end{cases} \tag{35}$$

as shown in figure 5(b) for a time after the *T*-wave and *L*-wave have separated.

This example illustrates some important characteristics of solutions to the coupled longitudinal–transverse wave equations when the string is driven primarily transversely. First, the longitudinal displacements, ξ_T, are of order A/w times smaller than the transverse displacements, η. Since conditions (22) require $A/w \ll 1$, it follows that the conventional assumption that longitudinal motion is much smaller than the transverse motion is valid when transverse motion obeys the linear transverse wave equation. This does not mean that longitudinal motion makes a negligible contribution to the location of energy however, because ξ' contributes to ε_p at first order while η' contributes at second order, and the reader can readily show using (B.2) that the isolated *T*-wave, (30) together with (33), carries *no* potential energy, while the potential energy conventional treatments would ascribe to the *T*-wave (i.e. $\int_{T-\text{wave}} \frac{1}{2}\tau_0\eta'^2\,\mathrm{d}X$) is in fact completely carried off by the *L*-wave given by (35). *And this is true regardless of how small A/w is made*, meaning that these results are *not* 'large amplitude effects'. This result also gives an indication as to why, as mentioned in the Introduction, such precursor waves have been found to contribute to the characteristic sound of struck piano strings [1–3] and plucked guitar strings [4].

It is also readily shown from (35), using constraint (22) and $\rho_0 c_T^2 = \tau_0$, that the kinetic energy associated with the *L*-wave, $\int_{L-\text{wave}} \frac{1}{2}\rho_0\dot{\xi}_L^2\,\mathrm{d}X$, is negligible compared to the potential energy given by $\int_{L-\text{wave}} \tau_0\xi_L'\,\mathrm{d}X$. The potential energy associated with the density $\frac{1}{2}(SY+\tau_0)\xi_L'^2$ from (B.1) is similarly negligible[8]. Thus, the energy of the *L*-wave is essentially just the potential energy normally associated with the *T*-wave. Since there exist 'rope-like' transverse waves [16] which just carry kinetic energy and whose transverse wave profile is governed by the linear transverse wave equation, this is why the linear transverse wave equation can still be valid even though a large amount of energy gets coupled to an *L*-wave.

It also follows from (35) that the net effect of the *L*-wave is to shift points of the string $-\pi^2A^2/(4w)$ units to the left, while the *T*-wave shifts the particles by the same amount to the right as shown in figure 5(c), and thus, after the passage of both waves, particles of the string are left in their equilibrium position. This behaviour was shown in figure 5 of [5] for a truncated Gaussian transverse wave pulse by numerically integrating the interactions of a chain of point masses joined by ideal Hookean springs, thus providing independent 'empirical' support for the theory above.

The fact that the *L*-wave *should* shift points of the string $\pi^2A^2/(4w)$ units to the left in the case under consideration can be seen to make physical sense because it is straightforward to show that the transverse pulse given by (30) *increases* the length of the string by exactly this amount (to the level of approximation being considered; i.e. $(1+\eta'^2)^{1/2} \approx 1+\frac{1}{2}\eta'^2$). That is, as far as the *L*-wave is concerned, moving the left-hand end of the string transversely so as to create a transverse wave which stretches the string by an amount of $\pi^2A^2/(4w)$ units is effectively the same as moving the left-hand end of the string $\pi^2A^2/(4w)$ to the left and leaving it there.

Other pulses that could be explored by students include a finite linear ramp (i.e. $\eta(0,t) = at, 0 \leqslant t \leqslant w$ and $\eta(0,t) = aw, t > w$ for some constant a) and a finite sinusoid (i.e. $\eta(0,t) = A\sin(\omega t), 0 \leqslant t \leqslant n\pi/\omega$, with n being an integer).

3.3. Travelling wave pulse on a finite string and the uniform tension approximation

The analysis in section 3.2 is something of an idealization as no account has been taken of the string boundary at $X = L_0$; it has been assumed that the string is long enough for the *L*-wave in figure 5 to fit on the string. However, the width of the *L*-wave is c_L/c_T times that of the *T*-wave, so this raises the possibility for a finite length string that when $c_L \gg c_T$, as occurs for metal strings such as in pianos, the width of the *L*-wave would be longer than that of the string and hence reflections of the *L*-wave off the $X = L_0$ boundary would need to be considered. An exact treatment considering reflections would be quite difficult, but a

[8] Except when trying to derive the longitudinal wave equation from the Euler–Lagrange equations.

simplifying approximation commonly made in the literature makes an approximate treatment possible, and this section will discuss this approximate treatment and its implications.When $c_L \gg c_T$, the longitudinal wave equation, (24), is commonly approximated in the literature by taking the $c_L \to \infty$ limit, which results in

$$0 \approx \xi'' + \eta'\eta'' = \frac{\partial}{\partial X}\left(\xi' + \frac{1}{2}\eta'^2\right). \tag{36}$$

(A more rigorous approach to deriving (36) is given in [26].) Equation (36) implies that

$$\xi'(X,t) + \tfrac{1}{2}\eta'^2(X,t) = f(t), \tag{37}$$

where $f(t)$ is some function of time only. The implication of (37) can be seen by noting from (A.4) that, to the order of approximation being considered, the local tension in the string is given by[9]

$$\tau \approx \tau_0\left[1 + \frac{c_L^2}{c_T^2}\left(\xi' + \frac{1}{2}\eta'^2\right)\right], \tag{38}$$

which implies that if $\xi' + \frac{1}{2}\eta'^2$ is independent of position, then the tension, and the potential energy density through (B.2), is always uniform across the entire string and is determined by the length of the string at every instant (e.g. [26, 27], section 5 of [30]). (Note the difference with the previous subsection: in the $c_L \to \infty$ limit, the tension is exactly uniform across the entire string with $\tau > \tau_0$, while in the general case considered in subsection 3.2, the tension is only exactly uniform over a pure T-wave with the tension in that case given by τ_0. The two cases are consistent because in the $c_L \to \infty$ limit, the L-wave cannot separate from the T-wave as the L-wave has an infinite width.)

Finally, the longitudinal displacement $\xi(X,t)$ can be found by integrating (37) from $X = 0$ at a given instant. Assuming $\xi(0,t) = 0$ and noting that if $\xi' + \frac{1}{2}\eta'^2$ is independent of position, then in (37), $f(t) = \xi'(0,t) + \frac{1}{2}\eta'^2(0,t)$, and so the result of this integral is thus

$$\xi(X,t) = \left[\xi'(0,t) + \frac{1}{2}\eta'^2(0,t)\right]X - \int_0^X \frac{1}{2}\eta'^2(X,t)\,\mathrm{d}X. \tag{39}$$

In (39), $\eta'(X,t)$ and hence $\eta'(0,t)$ are determined by the transverse boundary condition and transverse wave equation, (23), and the final unknown $\xi'(0,t)$ is determined by making (39) satisfy the right-hand boundary condition, such as $\xi(L_0,t) = 0$. An illustration of (39) for the case of the $\cos^2$ pulse considered in section 3.2 is shown in figure 6.

In figure 6, when the transverse wave has moved away from $X = 0$, (30) in (37) and requiring $\xi(L_0,t) = 0$ gives $\xi'(0,t) = \pi^2A^2/(4wL_0)$. This means from (B.2) that across the whole string, the potential energy density is uniform and given by

$$\varepsilon_p \approx \frac{\tau_0\pi^2A^2}{4wL_0} = \frac{\tau_0\Delta L}{L_0}, \tag{40}$$

where, as noted in subsection 3.2, $\Delta L = \pi^2A^2/(4w)$ is the amount the string is stretched by the entire $\cos^2$ pulse. The fact that ε_p should be given by (40) follows from the assumption that the waves are sufficiently small that the local tension does not differ significantly from τ_0, so the work done by this force in stretching the string by an amount ΔL is simply $\tau_0\Delta L$.

[9] Recall that for the usual linear transverse wave equation to remain valid, one requires $\tau \approx \tau_0$. Equation (38) indicates that for this condition to be satisfied as c_L/c_T increases in size, then either the spatial derivatives of the waves must correspondingly decrease in size or one needs to arrange for ξ' to be negative wherever there is a transverse wave. The latter condition can be met by decreasing the separation between the end supports of the string as the wave is created (see, for example, section VIID and appendices B2 and B3 of [5] and [6]).

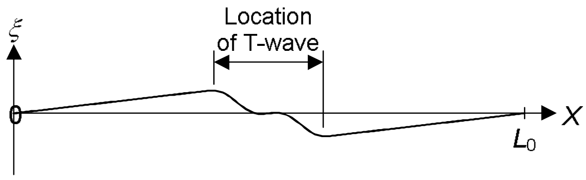

Figure 6. Longitudinal displacement for a $\cos^2$ travelling wave pulse given by (30) when the $c_L \to \infty$, uniform tension approximation has been made. For the case shown, the T-wave is located in the centre of the string, has a width $w = L_0/4$ and $A/w = 0.1$. The fact that $\xi\,(X, t)$ has a constant positive slope outside of the T-wave region makes it easy to see that the tension in the string is in fact constant there and higher than τ_0 (see (A.4)). Note that the vertical scale is greatly exaggerated compared to the horizontal scale for clarity.

As this total amount of energy is approximately uniformly distributed across the whole string, its density is just the total divided by the length between the end supports of the string, L_0.

3.4. Standing wave solution to the coupled wave equations

A fundamental transverse standing wave solution to (23) and (24) has been presented by Morse and Ingard ([18] section 14.3). These authors did not, however, indicate how they obtained their solution, so the purpose of this subsection is to sketch a possible approach.

Based on the results of subsection 3.2, one might guess that a transverse standing wave solution to (23) and (24) would result in the string always having a uniform (though time varying) tension. Thus, from subsection 3.3, one would guess that $\xi'' = -\eta'\eta''$. Substituting into this equation $\eta = A\sin(\pi X/L_0)\cos(\omega_T t)$ with $\omega_T = \pi c_T/L_0$ and L_0 being the length of the string when under tension τ_0 but no waves are present, and then integrating twice, one obtains

$$\xi = -\frac{\pi}{8}\frac{A^2}{L_0}\sin\left(\frac{2\pi X}{L_0}\right)\cos^2(\omega_T t) \tag{41}$$

as the guess, having used the boundary conditions to eliminate the integration functions of t. One finds however that (41) does not satisfy (24), although it nearly does, and it is not too hard to guess what additional term needs to be added to get the full solution which is ([18] section 14.3)

$$\xi = -\frac{\pi}{8}\frac{A^2}{L_0}\sin\left(\frac{2\pi X}{L_0}\right)\left(\cos^2(\omega_T t) - \frac{c_T^2}{2c_L^2}\right). \tag{42}$$

Note again that $O(|\xi'|) = O([A/L_0]^2) = O(\eta'^2)$ as assumed above.

4. Conclusions

Physics students are typically introduced to wave phenomena with a discussion of transverse waves on a taut string. As is often the case in physics, several idealizations are made to make the analysis as simple as possible while still capturing the essential physics. These idealizations are that the string is lossless and perfectly flexible, and that the amplitude of the transverse oscillations is sufficiently small that they are governed by a linear wave equation and that in comparison, longitudinal displacements are negligibly small. Based on these assumptions, authors of introductory physics texts derive formulas for the rate of energy propagation in a unidirectional travelling wave by assuming that all the energy in the wave travels at the same speed [9], or that the rate at which one segment of the string does work on the next is given, to a high degree of approximation, simply by the vertical component of the assumed constant tension multiplied by the vertical velocity of the string [10, 11]. The results of this and previous papers [6, 7], however, demonstrate that *for these derivations to be valid, a further idealization needs to be made, and that is that the string being modelled has to have a zero relaxed length so that* $Y = 0$ *and consequently that* $c_L = c_T$, for otherwise, longitudinal displacements have a far from negligible impact on the location and propagation of energy, *even for transverse waves with arbitrarily small amplitudes and rates of spatial variation*. Thus, when it comes to energy propagation, the strings discussed in introductory texts need to be limited to slinky springs[10] so that purely transverse motion is strictly possible and not just approximately true.

To understand the location and propagation of energy in the metal and nylon strings found in musical instruments, and in the strings used in physics demonstration apparatus though, the theory of this paper is needed. While the theory is too advanced for introductory classes, the results are probably sufficiently intuitive that they could be discussed in a qualitative way in such classes, and the examples worked section 3 would provide instructors with sufficient physical insight into the physical implications of the coupled transverse–longitudinal wave equations to be able to do so.

For intermediate to advanced classes in continuum mechanics or general wave phenomena though, the theory and examples in this paper could find direct use. The derivation of the general energy continuity equation in section 2.1, for example, is a simple and concrete application of the general continuum mechanics approach to deriving energy conservation laws for elastic bodies (e.g. [13]). This approach is an improvement on that given in section 5 of [7] in that it is more general, leads to a more easily interpretable form for the energy continuity equation, and as pointed out in section 2.2, does not suffer from the problem of not actually producing a unique answer.

For advanced undergraduate wave theory courses which go beyond the linear transverse wave equation to consider the consequences of loss, nonlinearity, flexural rigidity and so on, the results of section 3 have value in that they show that algebraic solutions to the linear transverse wave equation coupled to the longitudinal wave equation for some interesting wave types are possible and relatively simply obtained, *if the solution algorithm is known*. While parts of the algorithm for finding solutions when a string is driven at one end have been discussed previously in [5], section 3.1 appears to be the first exposition of a complete algorithm and demonstration that algebraic solutions are possible and readily obtained. Such algebraic solutions have value in allowing the validity of the approximations leading to (23) and (24) to be checked, and the origin and significance of the 'attack waves' reported in the musical acoustics literature [1–4] to be explored algebraically.

The final significant contribution of this paper is to present new arguments in the debate over whether potential energy density in strings is uniquely definable. It was noted that the precepts of general relativity require that in principle the answer must be that it can [24]. It was also demonstrated by using an external energy source to create a travelling wave pulse, that for total system energy to be conserved, the alternative wave potential energy density formula put forward in [8, 12, 19] needed an unphysical potential energy to be ascribed to the energy source and so could *not* be considered to be *physically* valid, even though one could make valid *mathematical* calculations with it.

[10] Elastic strings which can be stretched to many times their relaxed length might also approximately have the desired properties.

Appendix A. Formulas for the local tension in a taut string

The exact formula for the local tension, $\tau(X, t)$, in a linearly elastic, perfectly flexible string with Young's modulus Y, cross-sectional area S and equilibrium tension τ_0 is given by[11]

$$\tau = (\tau_0 + SY)|\mathbf{r}'| - SY. \tag{A.1}$$

Note that in many papers it is stated that $\tau \approx \tau_0 + SY(|\mathbf{r}'| - 1)$ (e.g. [3, 28–30]). These authors have taken $\tau_0 \ll SY$ to be true, or equivalently, that the equilibrium strain is small compared to unity.

To obtain approximate expressions from (A.1), a Taylor series expansion of $|\mathbf{r}'|$ is needed. With $\mathbf{r} = [X + \xi, \eta, \zeta]$,

$$|\mathbf{r}'| = (\mathbf{r}'^2)^{1/2} = (1 + 2\xi' + \xi'^2 + \eta'^2 + \zeta'^2)^{1/2} = (1 + \delta)^{1/2} \tag{A.2}$$

with $|\delta| \ll 1$. Since δ includes terms both linear and quadratic in ξ', a second-order Taylor series expansion of the right-hand side of (A.2) is needed since the second-order term will also include a term quadratic in ξ'. To second order, $(1 + \delta)^{1/2} \approx 1 + \frac{1}{2}\delta - \frac{1}{8}\delta^2$, and so assuming $O(|\xi'|) = O(\eta'^2) = O(\zeta'^2)$ and keeping terms only up to $O(\xi'^2)$, one obtains

$$|\mathbf{r}'| \approx 1 + \xi' + \tfrac{1}{2}(\eta'^2 + \zeta'^2) - \tfrac{1}{8}(\eta'^4 + \zeta'^4 + 2\eta'^2\zeta'^2) - \tfrac{1}{2}\xi'(\eta'^2 + \zeta'^2). \tag{A.3}$$

Substituting (A.3) into (A.1), one obtains after some rearrangement of terms and using $(SY + \tau_0)/\tau_0 = c_L^2/c_T^2$:

$$\tau \approx \tau_0 \left\{1 + \frac{c_L^2}{c_T^2}\left[\xi' + \frac{1}{2}(\eta'^2 + \zeta'^2) + \cdots\right]\right\}, \tag{A.4}$$

and thus inequalities (22) needed for the linear transverse wave equation to be valid can be seen to be a requirement that the magnitude of the fractional change in the local tension is always small in comparison to 1 (i.e. the usual assumption that $\tau \approx \tau_0$ is valid).

Two particular forms of (A.1) are of interest. First, for purely longitudinal motion, $\mathbf{r} = [X + \xi, 0, 0]$, so $|\mathbf{r}'| = 1 + \xi'$ (assuming $|\xi'| < 1$), and

$$\tau_{\text{long}} = \tau_0 + (\tau_0 + SY)\xi'. \tag{A.5}$$

Second, for a slinky spring for which $Y \approx 0$, (A.1) gives

$$\tau_{\text{slinky}} \approx \tau_0|\mathbf{r}'|. \tag{A.6}$$

Appendix B. Approximations to and special cases of the potential energy density

Using the Taylor series expansion (A.3) in (8), the following approximate expression for ε_p is obtained:

$$\varepsilon_p \approx \left[\tau_0 + \tfrac{1}{2}(SY + \tau_0)\xi'\right]\xi' + \tfrac{1}{2}\left[\tau_0 + SY\left(\xi' + \tfrac{1}{4}(\eta'^2 + \zeta'^2)\right)\right](\eta'^2 + \zeta'^2). \tag{B.1}$$

Note from (22) that when the conditions needed for the scalar wave equation to be valid are met,

$$\varepsilon_p \approx \tau_0\left[\xi' + \tfrac{1}{2}(\eta'^2 + \zeta'^2)\right]. \tag{B.2}$$

[11] See (25) in [27] after initially replacing τ_0 by $SY\varepsilon$ and multiplying through by ε, where in the notation of [27], $\varepsilon = (L_0 - L_r)/L_r$, is the equilibrium strain of the string, L_r is the relaxed length of the string and L_0 its equilibrium length when under tension τ_0.

Interestingly though, a Lagrangian density constructed from (B.2) does *not* yield the coupled wave equations (23) and (24) when substituted into the Euler–Lagrange equations, rather (B.1) is needed.

Again, two special cases are of interest. For purely longitudinal motion, (8) gives

$$\varepsilon_p(\text{longitudinal wave only}) = \tau_0\xi' + \tfrac{1}{2}(\tau_0 + SY)\xi'^2, \tag{B.3}$$

while for a slinky spring with $Y \approx 0$, (8) reduces to

$$\varepsilon_p(\text{slinky}) = \tfrac{1}{2}\tau_0(\mathbf{r}'^2 - 1) = \tfrac{1}{2}\tau_0(2\xi' + \xi'^2 + \eta'^2 + \zeta'^2). \tag{B.4}$$

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