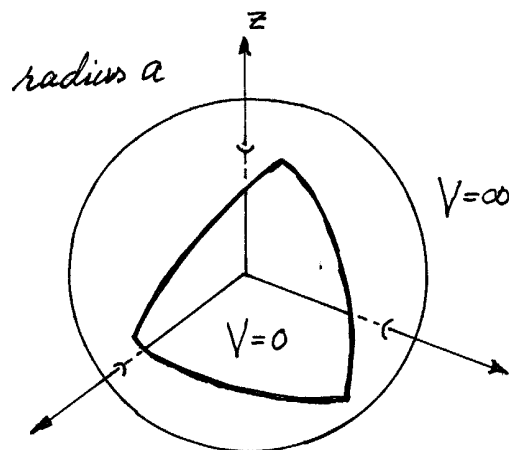<div align="center">Lecture notes 9</div>

9 Spherical box.   Systems with cylindrical symmetry

These notes treat the spherical box potential (section 9.1). The discussion of cylindrically symmetric systems in section 9.2 has not been a part of FY2045/TFY4250 (for some time).

9.1 Spherical box

[See 4.1.3 in Griffiths, and 7.3 in B&J.]

A very simple example of a spherically symmetric potential is the spherical box, with

$$V(r) = \begin{cases} 0 & \text{for} \quad r < a, \\ \infty & \text{for} \quad r > a. \end{cases}$$

The radial equation (T5.41) may then for $0 \leq r < a$ be written on the form

$$(kr)^2 \frac{d^2 R}{d(kr)^2} + 2(kr)\frac{dR}{d(kr)} + \left[(kr)^2 - l(l+1)\right] R(r) = 0, \qquad \text{(T9.1)}$$

where $k \equiv \sqrt{2mE/\hbar^2}$. This differential equation is known as the **spherical Bessel equation**.

Spherical Bessel and Neumann functions

For a given l the general solution of this equation is a linear combination of the **spherical Bessel function** $j_l(kr)$ and the **spherical Neumann function** $n_l(kr)$:

$$R(r) = A\, j_l(kr) + B\, n_l(kr).$$

These two types of functions are respectively regular and irregular (diverge) at the origin. The Neumann functions are therefore not relevant when we need solutions in an r-region which includes the origin, which is the case at hand. It can be shown that

$$j_l(z) = z^l \left(-\frac{1}{z}\frac{d}{dz} \right)^l \frac{\sin z}{z} \qquad \text{and} \qquad n_l(z) = -z^l \left(-\frac{1}{z}\frac{d}{dz} \right)^l \frac{\cos z}{z}. \qquad \text{(T9.2)}$$

Thus for $l = 0$ and 1 we have

$$j_0(z) = \frac{\sin z}{z}, \qquad\qquad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z},$$

$$n_0(z) = -\frac{\cos z}{z} \qquad \text{and} \qquad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z}.$$

Here we see that the Neumann functions diverge at the origin, while expansion for small z shows that the Bessel functions behave as

$$j_0(z) \approx 1 \qquad \text{and} \qquad j_1(z) \approx \frac{z}{3} \quad \text{for small } z.$$

For generel values of l it can be shown that $j_l(z) \propto z^l$ for small z. This shows that the behaviour for small r of the radial functions

$$R_l(r) = A_l j_l(kr) \qquad\qquad\qquad\qquad \text{(T9.3)}$$

for the spherical box agrees with the general formula (T5.44).

Energy quantization and wave functions

Because $V = \infty$ for $r > a$, we must require that $R_l(r) = 0$ for $r = a$. This leads to energy quantization. For $l = 0$ this is *very* simple. The continuity condition

$$R_0(a) = A j_0(ka) = A\,\frac{sinka}{ka} = 0$$

requires that ka is an integer multiple of π. Thus the allowed k-values are $k_n^{(0)} = \pi n/a$, so that the energies and the radial functions are

$$E_{n,l=0} = \frac{(\hbar k_n^{(0)})^2}{2m} = \frac{(\hbar \pi n)^2}{2ma^2} \qquad \text{and} \qquad R_{n,l=0}(r) = A_{n0}\frac{\sin(\pi n r/a)}{\pi n r/a}, \qquad n = 1, 2, \cdots.$$
$$\text{(T9.4)}$$

> These energies are the same as for the one-dimensional box of width a, and *that* is not at all surprising. This is because for $l = 0$ the effective potential is equal

to zero for $0 \leq r < a$, so that the "one-dimensional" radial equation (T5.42) for $u_0(r) = rR_0(r)$ takes the form

$$-\frac{\hbar^2}{2m}\frac{d^2u_0(r)}{dr^2} = E\,u_0(r), \quad \text{with} \quad u_0(0) = u_0(a) = 0.$$

This equation describes a one-dimensional box, and the solutions must all be of the type $\sin kr$ in order to satisfy the requirement $u_0(0) = 0$. The condition $u_0(a) = 0$ then requires that ka is equal to one of the <u>zeros</u> of the sine function, πn. This gives a set of solutions $u_{n0}(r) \propto \sin(\pi nr/a)$, which are precisely the solutions found above.

The figure shows the radial functions, that is, the spherical Bessel functions

$$j_0(k_n^{(0)}r) = \frac{\sin(\pi nr/a)}{\pi nr/a},$$

for $n = 1, 2, 3, 4$, corresponding to the radial quantum numbers $n_r = 0, 1, 2, 3$.
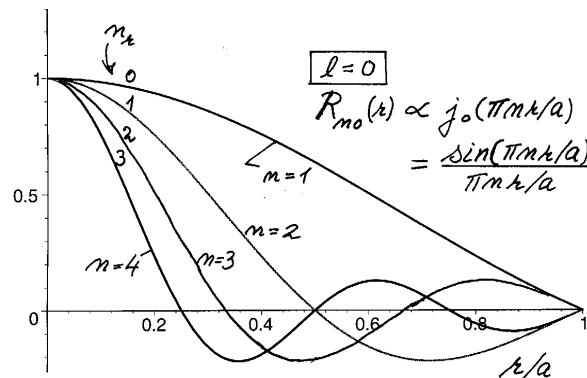
Note that all these s-waves differ from zero at the origin; for $l = 0$ there is no "centrifugal barrier", which would otherwise give a zero at the origin.

States with angular momentum

For $l \geq 1$ it is not quite as simple. Both $R_{nl}(r)$ and $u_{nl}(r) = rR_{nl}(r)$ now become more complicated, due to the effective potential which in this case consists of the centrifugal term:

$$V_{\text{eff}}^l(r) = \frac{\hbar^2 l(l+1)}{2mr^2}.$$

(See the figure on page 21 in Lecture notes 5.) For a given l the set $R_{nl}(r)$ of solutions is determined by the Bessel function $j_l(kr)$, which must have a zero at $r = a$. This requires that ka is one of the zeros of j_l, which we may denote by $\Pi_n^{(l)}$ (in analogy with the zeros πn of the function $R_0(r) = j_0(r) = \sin(kr)/r$):

$$R(a) = A\,j_l(ka) = 0 \quad \Longrightarrow \quad k_n^{(l)}a = \Pi_n^{(l)}.$$

The energies and the wave functions for angular momentum l thus are

$$E_{n,l} = \frac{(\hbar k_n^{(l)})^2}{2m} = \frac{(\hbar\Pi_n^{(l)})^2}{2ma^2} \quad \text{and} \quad \psi_{nlm}(r,\theta,\phi) = R_{n,l}(r)Y_{lm}(\theta,\phi)\,, \quad m = 0, \pm 1, \pm 2, \cdots$$

$$\text{(T9.5)}$$

$$\text{where} \qquad R_{n,l}(r) = A_{nl}\,j_l(\Pi_n^{(l)}r/a)\,, \qquad n = 1, 2, \cdots, .$$

The table gives some of the zeros $\Pi_n^{(l)}$ (given on page 467 in Abramowitz and Stegun).[1] Note that the radial quantum number is $n_r = n - 1$.

$$
\begin{array}{ccccc}
& j_0 & j_1 & j_2 & j_3 \\
n_r = 0 & \Pi_1^{(0)} = \pi & \Pi_1^{(1)} = 4.4934 & \Pi_1^{(2)} = 5.7635 & \Pi_1^{(3)} = 6.9879 \\
n_r = 1 & \Pi_2^{(0)} = \pi \cdot 2 & \Pi_2^{(1)} = 7.7253 & \Pi_2^{(2)} = 9.0950 & \Pi_2^{(3)} = 10.4171 \\
n_r = 2 & \Pi_3^{(0)} = \pi \cdot 3 & \Pi_3^{(1)} = 10.9041 & \Pi_3^{(2)} = 12.3229 & \Pi_3^{(3)} = 13.6980
\end{array}
\tag{T9.6}
$$

A small exercise: Show that the zeros $\Pi_n^{(1)}$ of $j_1(z)$ are determined by the condition $\tan \Pi_n^{(1)} = \Pi_n^{(1)}$, and check that the zeros in the table satisfy this condition.
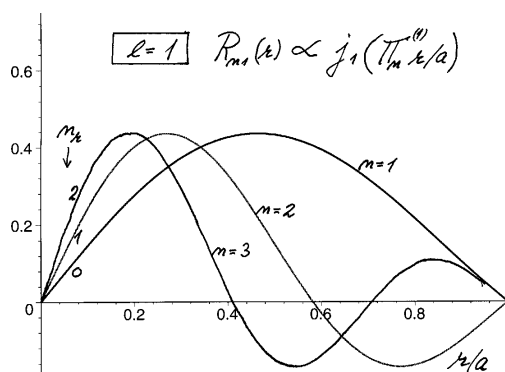
The figure shows the radial functions $R_{n1}(r) \propto j_1(\Pi_n^{(1)} r/a)$ for $l = 1$ and $n = 1, 2, 3$, corresponding to the radial quantum numbers $n_r = 0, 1, 2$. These radial functions all behave as $\sim r$ for small r. For higher angular momenta (l) the centrifugal term causes a stronger suppression of the radial functions near the origin; as mentioned above they behave as $\sim r^l$ for small r.

Another small exercise: Use the above table to find the quantum numbers for the first excited levels. Find the (degree of) degeneracy for these levels. Suppose that we put a small number of non-interacting fermions with spin $\frac{1}{2}$ into the box. How many fermions are allowed in each energy level?
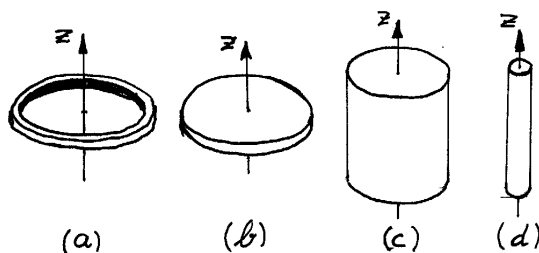
9.2 Potentials with cylindrical symmetry ***[2]

9.2.a "Two-dimensional" systems

(See 5.3 in Hemmer)

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[1]These zeros, and vast amounts of other information both on Bessel functions and other important **special functions**, are available in a standard handbook by M. Abramowitz and I.A. Stegun; *Handbook of Mathematical Functions*.
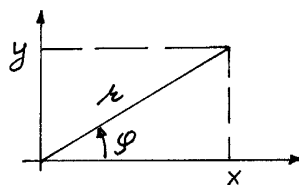
[2]Not compulsory in FY2045/TFY4250.

The figure shows four "box" potentials, all of which are symmetric with respect to the z-axis. In (a) the "one-dimensional" box on page 4 in Lecture notes 8 has been bent around into a circular ring. In (b) this ring has been changed to a circular disk. In (c), the dimension in the z-direction of this disk has been increased, and is no longer small. In (d) the diameter of the cylinder (c) has been made small, so that we have what might be called (a box version of) a quantum wire. In all these cases, the motion in the z-direction is in a way trivial, being as in a one-dimensional box. In (a) and (b), the z-degree of freedom is excited only at fairly high energies.

Polar coordinates for two-dimensional systems

If we neglect the motion in the z-direction, we are left with an essentially two-dimensional system with a rotational symmetry with respect to the z-axis. For such two-dimensional systems with rotational symmetry — whether they are box potentials, more realistic wells or more complicated potentials — it is convenient to use polar coordinates in the plane, as described in section 5.3 in Hemmer:

$$x = r\cos\phi, \qquad\qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\phi, \qquad\qquad \tan\phi = y/x,$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\phi\,, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\phi\,, \qquad \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r}\,, \qquad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r}.$$

Using the chain rule we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x} = \cos\phi\,\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\,\frac{\partial}{\partial \phi}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y} = \sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\,\frac{\partial}{\partial \phi}.$$

Using these relations it is easy to show that

$$\widehat{L}_z = \frac{\hbar}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = \frac{\hbar}{i}\,\frac{\partial}{\partial \phi}$$

(as in spherical coordinates). With a little more labour you will also find that the Laplacian in these coordinates is [3]

$$\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{\widehat{L}_z^2}{\hbar^2 r^2}. \tag{T9.7}$$

Commuting operators and simultaneous eigenfunctions

In this subsection we denote the particle mass by μ, to avoid mix-up with the magnetic quantum number m. The Hamiltonian then takes the form

$$\widehat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 + V(r) = -\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\widehat{L}_z^2}{2\mu r^2} + V(r). \tag{T9.8}$$

Here we can identify the first term on the right as the radial part $(\widehat{K}_r)$ of the kinetisc-energy operator $\widehat{K}$ for the two-dimensional motion. The second term, the operator $\widehat{K}_L = \widehat{L}_z^2/(2\mu r^2)$, obviously is the rotational part of the kinetic-energy operator. With a rotationally symmetric potential $V(r)$ it is easy to see that $\widehat{H}$ commutes with $\widehat{L}_z$. [4] This means that it is possible to find simultaneous eigenfunctions $\psi(r,\phi)$ to $\widehat{H}$ and $\widehat{L}_z$: [5]

$$\widehat{H}\psi = E\psi,$$
$$\widehat{L}_z\psi = m\hbar\,\psi.$$

Here, the dimensionless quantum number m gives the eigenvalue of $\widehat{L}_z$. (That's why we chose to call the particle mass μ.)

The eigenvalue equation for $\widehat{L}_z$

For fixed r the eigenvalue equation for $\widehat{L}_z$ takes the form

$$\frac{d\psi}{\psi} = im\,d\phi.$$

Integration over ϕ then gives $\ln\psi = \ln R(r) + im\phi$, where the constant of integration $R(r)$ is independent of ϕ, but may be an arbitrary function of r, such that

$$\psi(r,\phi) = R(r)\,e^{im\phi}.$$

---

[3]By adding $\partial^2/\partial z^2$ you get the Laplacian in cylindrical coordinates; see Rottmann.

[4]In the three-dimensional case, with $V = V(r,z)$, that is, rotational symmetry with respect to the z-axis, it should be noted that $\widehat{H}$ commutes with $\widehat{L}_z$, but not with the other components of $\widehat{\mathbf{L}}$, and not with $\widehat{\mathbf{L}}^2$.

[5]According to equation (4.19) in Hemmer, the fact that $\hat{L}_z$ commutes with $\widehat{H}$ also means that

$$\frac{d}{dt}\langle L_z \rangle = \frac{i}{\hbar}\left\langle [\widehat{H}, \widehat{L}_z] \right\rangle = 0,$$

implying that $\langle L_z \rangle$ is time independent for an arbitrary state in the potential $V(r)$. We then say that L_z is a quantum-mechanical **constant of motion**. See section 4.3 in Hemmer.

Here, the quantum number m cannot be quite arbitrary. In order for ψ to be continuous, we must namely have

$$\psi(r,0) = \psi(r,2\pi) \quad \Longrightarrow \quad e^{2im\pi} = 1 \quad \Longrightarrow \quad m = 0, \pm1, \pm2, \cdots. \tag{T9.9}$$

Note that the continuity condition above is the same as that used in equation (T8.8) in Lecture notes 10. Note also that the resulting *quantization* of the angular momentum L_z simply stated follows from the fact that the wavefunction must "bite its own tail" at $\phi = 0$, which also corresponds to $\phi = 2\pi$. The quantization thus is a consequence of the angular space being "compact".

The radial equation

Inserting $\psi = R(r)e^{im\phi}$ into the eigenvalue equation $\widehat{H}\psi = E\psi$ and using

$$\hat{L}_z^2\psi = R(r)\hat{L}_z^2 e^{im\phi} = \hbar^2 m^2 R(r)e^{im\phi} \tag{T9.10}$$

and (T9.9), we now find that the radial function $R(r)$ must satisfy the **radial equation**

$$\boxed{-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)R(r) + \left[V(r) + \frac{\hbar^2 m^2}{2\mu r^2}\right]R(r) = ER(r).} \tag{T9.11}$$

Here we notice that the potential $V(r)$ occurs together with the term

$$\frac{\hbar^2 m^2}{2\mu r^2} = \frac{L_z^2}{2\mu r^2}.$$

This is a centrifugal term corresponding to the centrifugal force that can be associated with the angular momentum L_z. The centrifugal term and the potential together form what we might call an **effektive potential**

$$V_{\text{eff}}^{|m|}(r) = V(r) + \frac{\hbar^2 m^2}{2\mu r^2}, \qquad |m| = 0, 1, 2, \cdots, \tag{T9.12}$$

where we note that the last term is repulsive, and more so the higher $|m|$ is. Thus, for each value of $|m|$ we have a radial equation , with a set of radial solutions $R^{(m)}(r)$ which can be characterized by $|m|$ and the number of zeros n_r (the so-called **radial** quantum number). Since the corresponding energies do not depend on the sign of m we have degeneracy 2 for $|m| \geq 1$.

Since the curvature of the radial function increases with the number of zeros, the energies E will in general increase with the radial quantum number n_r for fixed $|m|$. The energies will also increase with $|m|$ for fixed n_r, because the positive (and repulsive) centrifugal term increases with increasing $|m|$. [6] This is illustrated in the i example below.

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[6]These conclusions are valid even if we have not written the radial equation on one-dimensional form. That can be done easily. By introducing the new function $v(r)$ through $R(r) = v(r)/\sqrt{r}$, one finds instead of the radial equation (T9.11) a radial equation for the function $v(r)$ in which the term $r^{-1}d/dr..$ goes away, while the term $\hbar^2 m^2/(2\mu r^2)$ is replaced by $\hbar^2(m^2 - 1/4)/(2\mu r^2)$.
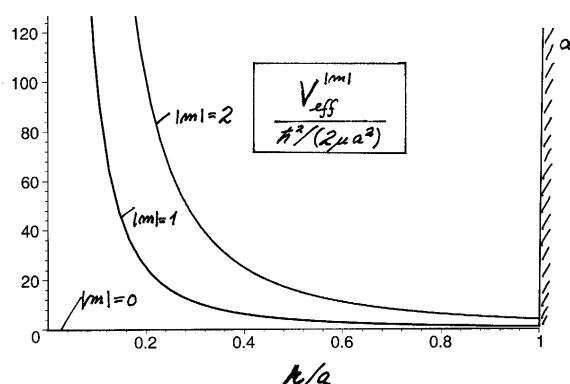
9.2.b Circular two-dimensional box
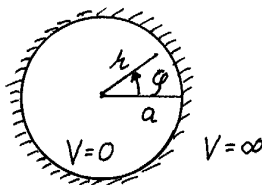
Here we assume that the particle is forced to move on a circular area of radius a. This corresponds to a two-dimensional circular box potential

$$V(r) = \begin{cases} 0 & \text{for} \quad 0 < r < a, \\ \infty & \text{for} \quad r > a. \end{cases}$$

The figure shows the effective potential in (T9.11) for $|m| = 0, 1$ and 2.
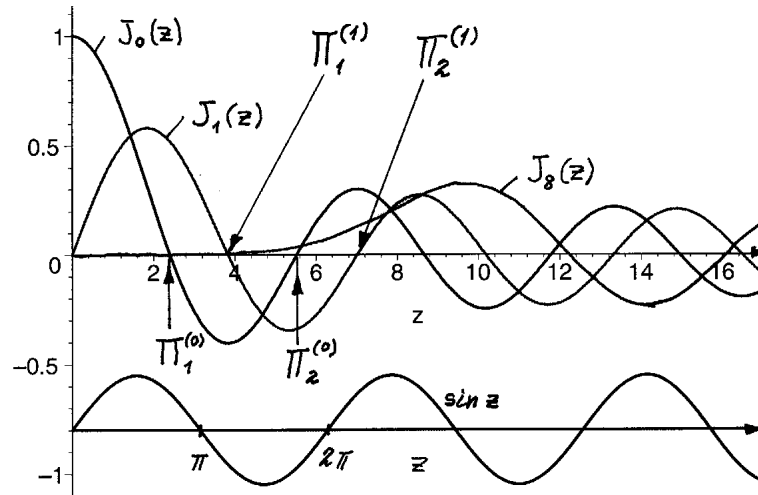

With

$$E \equiv \frac{\hbar^2 k^2}{2\mu}$$

we can then write the radial equation on the form

$$(kr)^2 \frac{d^2 R}{d(kr)^2} + kr \frac{dR}{d(kr)} + \left[(kr)^2 - m^2 \right] R = 0, \qquad (0 < r < a), \qquad \text{(T9.13)}$$

with the boundary condition $R(a) = 0$, since the wavefunction must be continuous and equal to zero for $r > a$. It is this radial equation together with the boundary condition that determine the allowed values of k and hence the energy eigenvalues, for each value of $|m|$. From the diagram above and the discussion in subsection **9.2.a** above we must expect to find the ground state for $m = 0$. Furthermore, we will find a set of radial solutions for each value of $|m|$.

This differential equation is a standard equation in applied mathematics, and is known as the **Bessel equation**. The solutions, the so-called cylinder functions, are widely used in physics and technology.

For each value of m (≥ 0) the equation has two independent solutions. These can be chosen such that one of them, the **Bessel function** $J_m(kr)$, is regular (finite) at the origin, which we must here require for our radial functions. The other solution, the **Neumann function** $N_m(kr)$, is *irregular* at the origin (goes infinite as $r \to 0$) and can therefore not be used in a region which includes the origin, which is the case for the present box potential.

The figure shows $J_0(z)$ (relevant for $m = 0$), $J_1(z)$ (relevant for $m = \pm 1$) and $J_8(z)$ (relevant for $m = \pm 8$). Under these curves we have for comparison included $\sin z$, for which the zeros are at $z = \pi n$. The zeros of the Bessel function $J_m(z)$ are called $\Pi_n^{(m)}$, in analogy with the similar notation in section 9.1. Some of these zeros are given in the table:[7]

$$
\begin{array}{cccc}
J_0 & J_1 & J_2 & J_3 \\
\Pi_1^{(0)} = 2.4048 & \Pi_1^{(1)} = 3.8317 & \Pi_1^{(2)} = 5.1356 & \Pi_1^{(3)} = 6.3802 \\
\Pi_2^{(0)} = 5.5201 & \Pi_2^{(1)} = 7.0156 & \Pi_2^{(2)} = 8.4172 & \Pi_2^{(3)} = 9.7610 \\
\Pi_3^{(0)} = 8.6537 & \Pi_3^{(1)} = 10.1735 & \Pi_3^{(2)} = 11.6198 & \Pi_3^{(3)} = 13.0152
\end{array}
\qquad \text{(T9.14)}
$$

For an ordinary box in one dimension, you will recall that the ground state corresponds to the part of the solution $\sin kx$ lying between the origin and the first zero. In analogy with this and the corresponding discussion in section 9.1, we must for the two-dimensional circular box require that the radial function $R^{(m)}(r) = J_{|m|}(kr)$ has a zero for $r = a$. Taking $m = 0$ as an example, we get

$$
k_n^{(0)} a = \Pi_n^{(0)} \qquad \text{and} \qquad E_n^{(0)} = \frac{(\hbar k_n^{(0)})^2}{2\mu} = \frac{(\hbar \Pi_n^{(0)})^2}{2\mu a^2}. \qquad \text{(T9.15)}
$$

The lowest of these energies corresponds to the lowest zero in the table, $\Pi_1^{(0)} = 2.4048$ (for $n = 1$). The corresponding radial function is the part of the Bessel function J_0 lying between 0 and $\Pi_1^{(0)}$:

$$
R_1^{(0)}(r) = J_0(k_1^{(0)} r) = J_0(\Pi_1^{(0)} r/a).
$$

This solution is shown in the diagram below, where we have also included the solutions for $n = 2$ and $n = 3$, with respectively one and two zeros in the interval $0 < r < a$. The energies of these states are obtained on inserting the values for $\Pi_n^{(0)}$ given in the table into (T9.15). As the table shows, the energy increases with the number of zeros $(n - 1)$; more zeros in the radial function means more curvature and higher kinetic energy.

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[7]These zeros, and lots of other information both on Bessel functions and other important **special functions**, are available in M. Abramowitz and I.A. Stegun; *Handbook of Mathematical Functions.*
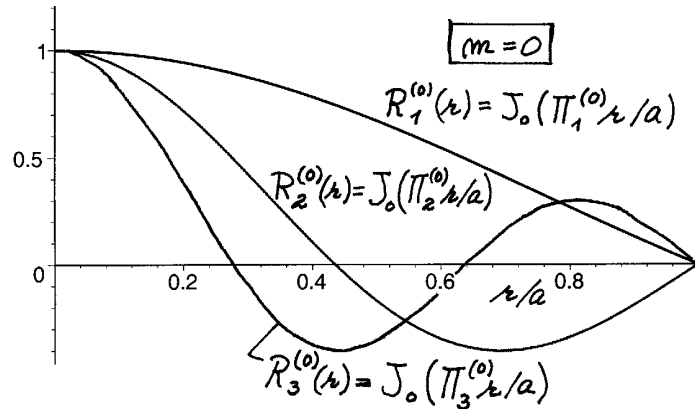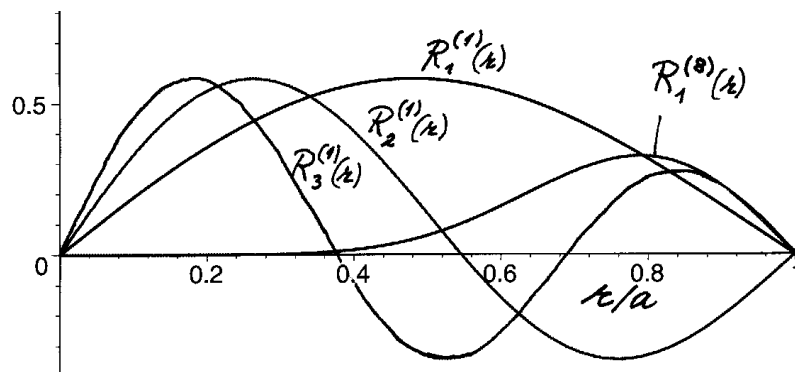
These radial functions are all determined by the Bessel function J_0:

$$R_n^{(0)}(r) = J_0(k_n^{(0)}r) = J_0(\Pi_n^{(0)}r/a). \tag{T9.16}$$

For $|m| \geq 1$ we proceed in the same manner: The energies and the corresponding radial functions follow from the condition $k_n^{(m)}a = \Pi_n^{(m)}$, which gives

$$E_n^{(m)} = \frac{(\hbar k_n^{(m)})^2}{2\mu} = \frac{(\hbar\Pi_n^{(m)})^2}{2\mu a^2} \qquad \text{and} \qquad R_n^{(m)}(r) = J_m(\Pi_n^{(m)}r/a). \tag{T9.17}$$

The figure below shows the radial functions $R_n^{(1)}(r)$ for $|m| = 1$ and $n = 1, 2$ and 3.

These radial functions for $|m| = 1$ all go to zero approximately as r for small r (contrary to the radial functions for $m = 0$ which have a maximum for $r = 0$). This is due to the centrifugal term for $|m| = 1$. In this figure we have also included the radial function $R_1^{(8)}(r) = J_8(\Pi_1^{(8)}r/a)$ for $|m| = 8$. Here the centrifugal term is so strongly repulsive that the radial function is strongly suppressed for small r. (It goes as r^8 close to the origin.)

Apart from normalization constants the complete eigenfunctions and energies are given by

$$\psi_n^{(m)} = e^{im\phi} R_n^{(|m|)}(r) = e^{im\phi} J_{|m|}(\Pi_n^{(|m|)}r/a) \qquad \text{and} \qquad E_n^{(m)} = \frac{(\hbar\Pi_n^{(|m|)})^2}{2\mu a^2}, \tag{T9.18}$$

where $m = \pm|m|$ and $n = 1, 2, \cdots$. We note that the probability densities are rotationally symmetric, given by the squares of the radial function.

From the figures above we see that the number of zeros (of the radial functions) for $0 < r < a$ are given by $n - 1$. This is what we call the radial quantum number,

$$n_r \equiv n - 1.$$

For $n_r \geq 1$ $(n \geq 2)$, we have excitatiuon of the "radial degree of freedom". In analogy, $|m| \geq 1$ corresponds to excitation of the "rotational degree of freedom".

In the ground state, with $m = 0$ and $n = 1$, none of these degrees of freedom are of course excited. The eigenfunction $\psi_1^{(0)} = R_1^{(0)}(r)$ is then ϕ-independent (no angular momentum L_z), and the radial function has no zero for $0 < r < a$, corresonding to minimal curvature in the radial direction.

The question now is whether the first excited level corresponds to an excitation in the angular or radial direction? The answer is given by the table, which shows that the next to the lowest energy level is $E_1^{(1)} = (\hbar\Pi_1^{(1)})^2/(2\mu a^2)$, with $\Pi_1^{(1)} = 3.8317$. The two states corresponding to this level have radial quantum number $n_r = n - 1 = 0$ and angular-momentum quantum number $m = \pm 1$ (that is, $L_z = \pm\hbar$). So here it is the rotational motion that is excited, giving ϕ-dependent eigenfunctions, proportional to $e^{\pm i\phi}$.

The table also shows that the second excited level is $E_1^{(2)} = (\hbar\Pi_1^{(2)})^2/(2\mu a^2)$, with $\Pi_1^{(2)} = 5.1356$. Her we thus have two states with $L_z = \pm 2\hbar$, and the energy increase is caused by a faster angular variation, through the two factors $e^{\pm 2i\phi}$.

The radial excitation enters the picure first at the third excited level, which is $E_2^{(0)} = (\hbar\Pi_2^{(0)})^2/(2\mu a^2)$, with $\Pi_2^{(0)} = 5.5201$. Here, we have a state with no rotation, $\psi_2^{(0)} = R_2^{(0)}(r)$, with $m = 0$ and $n_r = n - 1 = 1$. The radial excitation shows up by $R_2^{(0)}(r)$ having a zero for $r \approx 0.44\,a$ (see the figure). The increase in the radial curvature gives a higher energy than for the states belonging to the first and second excited levels. Note that the zero in the radial function results in a circular *nodal line* in the eigenfunction $\psi(r, \phi)$.

Part of the "moral" of this discussion is that the sequence of energy levels is determined by the zeros $\Pi_n^{(m)}$ of the Bessel functions; cf the table. If we imagine that this two-dimensional box is "filled" with a number of identical non-interacting spin-$\frac{1}{2}$ fermions, then the "filling sequence" is determined by the table, with two fermions per spatial state $\psi_n^{(m)}(r, \phi)$. This is analoguous to the filling sequence for atomic electrons. The particle density for such a many-fermion state will be rotationally symmetric, since it is a sum of symmetric probability densities

$$|\psi_n^{(m)}(r, \phi)|^2 = \left[R_n^{(m)}(r)\right]^2.$$

The figure shows a scanning-tunneling microscope picture of a surface of copper. On this surface a "quantum corral" of 48 iron atoms has been deposited. The picture shows clearly how the resulting density of the surface electrons inside the corral is rotationally symmetric.
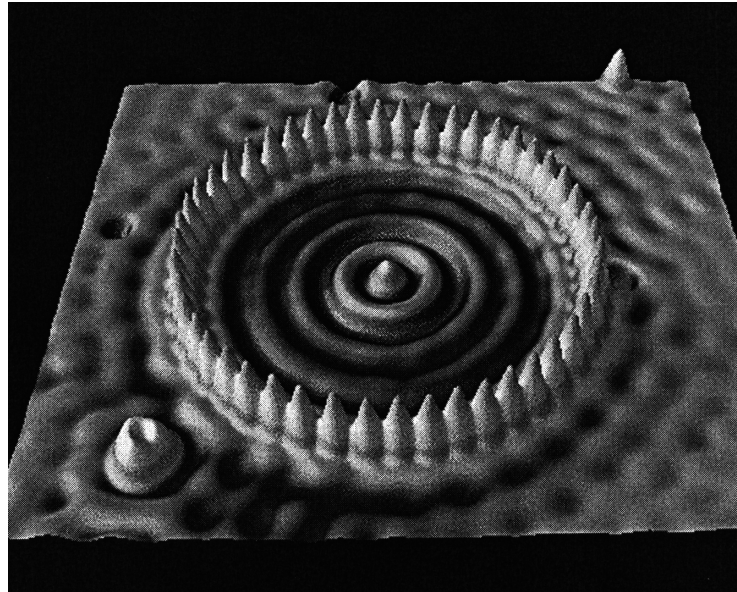
The figure below shows an attempt to describe the observed electron density using states of a cylindrical box.