2. Variational principles and Lagrange's equations

2.1 Hamilton's principle

We derived Lagrange's equations from D'Alembert's differential principle, by considering small virtual displacements from a given state of the system.

Next, we will derive Lagrange's equations from an integral (global) principle, by considering small variations in the total movement of the system.

The state of the system at a time t is given by a point in a configuration space with axes corresponding to the generalized coordinates $q_1 \dots q_n$ $(n = 3N - k)$. The movement of the system is then described by a curve in this space.

Hamilton's principle: The system moves from t_1 to t_2 such that the action

$$I = \int_{t_1}^{t_2} L \, dt \qquad (L = T - V = \text{Lagrangian})$$

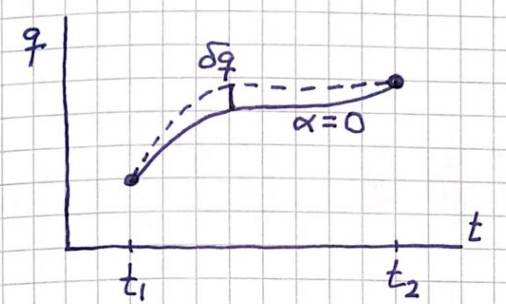
has a stationary value for the actual path of the motion, i.e.,

$$\delta I = \delta \int_{t_1}^{t_2} L(q_1 \cdots q_n ; \dot{q}_1 \cdots \dot{q}_n ; t) \, dt = 0$$

(for fixed t_1 and t_2)

2.3 Lagrange's equations from Hamilton's principle

Assume one degree of freedom $q = q(t)$



Paths near actual path are parametrized with parameter α.
Actual path : $\alpha = 0$, $\delta I = 0$

$$\Rightarrow \quad q(t,\alpha) = q(t,0) + \underbrace{\alpha\,\eta(t)}_{\delta q(t)} \qquad (\eta(t_1) = \eta(t_2) = 0)$$

(otherwise arbitrary $\eta(t)$)

$$I(\alpha) = \int_{t_1}^{t_2} L\,[\,q(t,\alpha),\ \dot{q}(t,\alpha),\ t\,]\,dt$$

Virtual variations at fixed time t:

$$\delta q = \left(\frac{\partial q}{\partial \alpha}\right)_0 d\alpha$$

$$\delta \dot{q} = \left(\frac{\partial \dot{q}}{\partial \alpha}\right)_0 d\alpha$$

Variation of the action:

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q} \right] dt$$

In $\delta \dot{q} = \delta \frac{d}{dt} q$, δ and $\frac{d}{dt}$ may be interchanged:

$$\delta \dot{q} = \left(\frac{\partial \dot{q}}{\partial \alpha}\right)_0 d\alpha = \frac{\partial^2 q}{\partial \alpha \partial t}\, d\alpha = \dot{\eta}\, d\alpha$$

$$\frac{d}{dt}\delta q = \frac{d}{dt}\left(\frac{\partial q}{\partial \alpha}\right)_0 d\alpha = \frac{d}{dt}\eta\, d\alpha = \dot{\eta}\, d\alpha$$

$$\Rightarrow \quad \delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\,\delta q + \frac{\partial L}{\partial \dot{q}}\,\frac{d}{dt}\,\delta q \right] dt$$

Integration by parts on 2. term:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \, dt = \underbrace{\Big|_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \delta q}_{=0 \text{ since } \delta q = 0 \text{ in } t_1 \text{ and } t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \, dt$$

$$\Rightarrow \delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right] \delta q \, dt = 0$$

Since δq is arbitrary: $\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$

Generalization to n degrees of freedom:

$$\delta I = \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \, dt = 0$$

$$\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad ; \quad i = 1, \dots, n$$

Valid when q_i are independent, i.e., with holonomic constraints.

Valid for conservative systems, $V = V(q_i)$, but also for non-conservative systems when the gen. forces can be expressed as

$$Q_i = -\frac{\partial U}{\partial q_i} + \frac{d}{dt} \frac{\partial U}{\partial \dot{q}_i} \qquad (L = T - U)$$

I.e., valid for socalled monogenic systems.

2.2 Calculus of variations

Variational principle not limited to the Lagrangian and corresponding action integral.

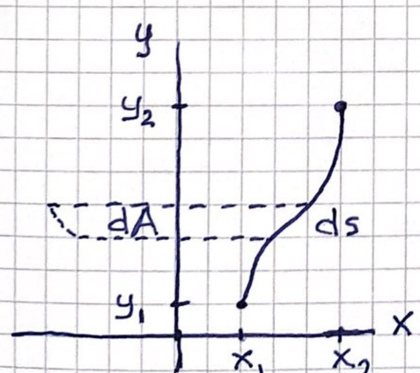
Consider a curve $y = y(x)$ between $y_1 = y(x_1)$ and $y_2 = y(x_2)$; $y' = dy/dx$. We want to find a stationary value (minimum or maximum) of the integral

$$I = \int_{x_1}^{x_2} f(y, y', x)\, dx$$

where f is a function defined on the curve $y(x)$. We adopt the derivation on p. 20-21, with $t \to x$, $q \to y$, $\dot{q} \to y'$ and $L(q, \dot{q}, t) \to f(y, y', x)$, and find:

$$\frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0 \qquad \text{(Euler's equations)}$$

Ex 1: Minimum surface of revolution



Which curve $y(x)$ between (x_1, y_1) and (x_2, y_2) gives minimum surface of revolution?

$$dA = 2\pi x \cdot ds = 2\pi x \cdot \sqrt{dx^2 + dy^2} = 2\pi x \sqrt{1 + y'^2}\, dx$$

$$\Rightarrow A = 2\pi \int_{x_1}^{x_2} x \sqrt{1 + y'^2}\, dx \quad \Rightarrow f(y, y', x) = x\sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial y'} = \frac{x y'}{\sqrt{1+y'^2}} \Rightarrow \frac{d}{dx} \frac{x y'}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow \frac{x y'}{\sqrt{1+y'^2}} = a \quad (= \text{const.}) \quad \Rightarrow \ldots \Rightarrow \frac{dy}{dx} = \frac{a}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \quad y = a \int \frac{dx}{\sqrt{x^2-a^2}}$$

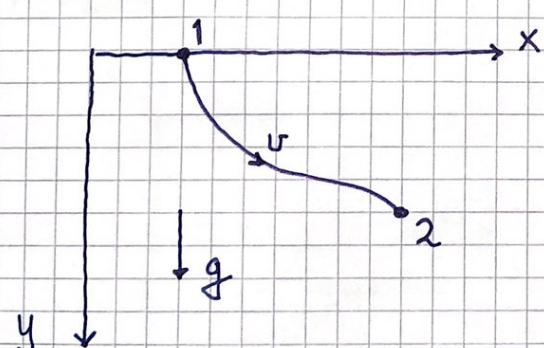
Subst. $x = a \cosh z \quad \Rightarrow dx = a \sinh z \, dz$

$$x^2 - a^2 = a^2 \sinh^2 z$$

$$\Rightarrow y = a \int dz = az + b = a \operatorname{arcosh}(x/a) + b$$

or $x = a \cosh\left(\frac{y-b}{a}\right)$, where a and b are determined by $y(x_1) = y_1$ and $y(x_2) = y_2$

Ex 2: The brachistochrone problem



Which curve from 1 to 2 gives shortest time? Assume $v(1) = 0$ and energy conservation.

$$dt = ds/v \quad ; \quad ds = dx \cdot \sqrt{1+y'^2}$$

$$\frac{1}{2} m v^2 = mgy \Rightarrow v = \sqrt{2gy}$$

$$\Rightarrow t_{12} = \int_1^2 dt = \frac{1}{\sqrt{2g}} \int_1^2 f(y, y', x)\, dx \quad ; \quad f = \sqrt{\frac{1+y'^2}{y}}$$

$$\frac{\partial f}{\partial y} = -\frac{\sqrt{1+y'^2}}{2y^{3/2}} \quad ; \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{y}\sqrt{1+y'^2}}$$

$$\Rightarrow \quad \frac{d}{dx} \frac{y'}{\sqrt{y}\sqrt{1+y'^2}} + \frac{\sqrt{1+y'^2}}{2y^{3/2}} = 0$$

Carry out d/dx using product and chain rule.
Cancel common factor $y^{-1/2}(1+y'^2)^{-1/2}$.
Multiply with y'. Use rules for logarithm.

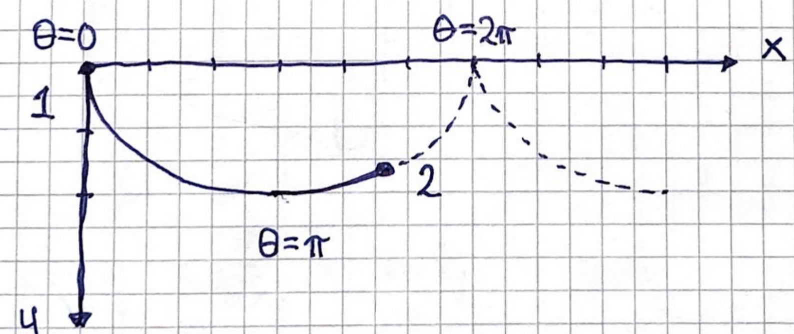
$$\Rightarrow \frac{d}{dx}\{\ln[y(1+y'^2)]\} = 0 \quad \Rightarrow y(1+y'^2) = 2k \quad (= \text{const.})$$

Parametric solution: $x = k(\theta - \sin\theta)$; $y = k(1-\cos\theta)$

[Check: $\frac{dy}{dx} = \frac{k\sin\theta\, d\theta}{k(1-\cos\theta)d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$$\Rightarrow 1 + y'^2 = \ldots = \frac{2}{1-\cos\theta} \Rightarrow y(1+y'^2) = 2k \text{ ; OK}]$$

Assume $x_1 = y_1 = 0$:



Cycloid (same as path followed by point on rolling wheel)

2.6 Conservation laws and symmetry properties

Consider first a system of point masses m_i in a potential V only dependent on positions. Then:

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_j \frac{1}{2} m_j \left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2\right) = m_i \dot{x}_i = p_{ix}$$

With generalized coordinates q_i, one defines the generalized (or canonical, or conjugate) momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

If the potential depends on velocities, $U = U(q_i, \dot{q}_i)$, canonical momentum and mechanical momentum are different.

Ex: Particle in E.M. field

$$U = q\phi - q\vec{A}\cdot\vec{v} = q\phi - qA_i\dot{x}_i$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}_i\dot{x}_i$$

$$L = T - U$$

$$\Rightarrow \underbrace{p_j}_{\text{canonical mom.}} = \frac{\partial L}{\partial \dot{x}_j} = \underbrace{m\dot{x}_j}_{\text{mechanical mom.}} + qA_j$$

Cyclic coord. q_i if L is independent of q_i

Then: $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i = 0 \Rightarrow p_i = \text{const.}$

$\Rightarrow$ The canonical momentum corresponding to a cyclic coordinate is conserved.

Ex: Particle in EM field with $\Phi, \vec{A}$ indep. of x

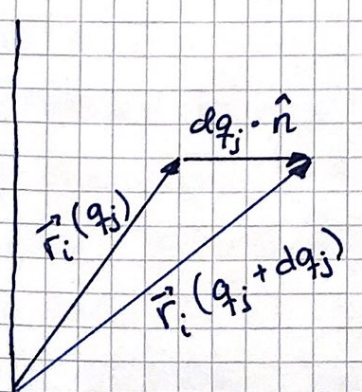
$\Rightarrow L$ indep. of x $\Rightarrow$ x is cyclic coord.

$\Rightarrow p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x = \text{const.}$

(but mech. mom. $m\dot{x}$ is not conserved)

Let us take a closer look at conservation laws related to translation and rotation, as well as conservation of energy.

Translation:
Consider gen. coord q_j such that dq_j implies translation of the whole system in a direction $\hat{n}$



$$\frac{\partial \vec{r}_i}{\partial q_j} = \lim_{dq_j \to 0} \frac{\vec{r}_i(q_j + dq_j) - \vec{r}_i(q_j)}{dq_j}$$

$$= \frac{dq_j \cdot \hat{n}}{dq_j} = \hat{n}$$

(for all particles i)

Assume conservative system, $V = V(q)$.
Clearly, $\partial T/\partial q_j = 0$. (cf. Ex 1a on p. 11 ; velocities, and hence T, unaffected by shifting the origin)

Lagrange eqn: $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = Q_j$ or: $\dot{p}_j = Q_j$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\underset{(p.10)}{=} \sum_i m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\sum_i m_i \vec{v}_i\right)\cdot \hat{n} = \vec{P}\cdot\hat{n}$$

= component of total lin. mom. $\vec{P}$ along $\hat{n}$

$$Q_j \overset{(p.10)}{=} \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\sum_i \vec{F}_i\right)\cdot\hat{n} = \vec{F}\cdot\hat{n}$$

= comp. of total force $\vec{F}$ along $\hat{n}$

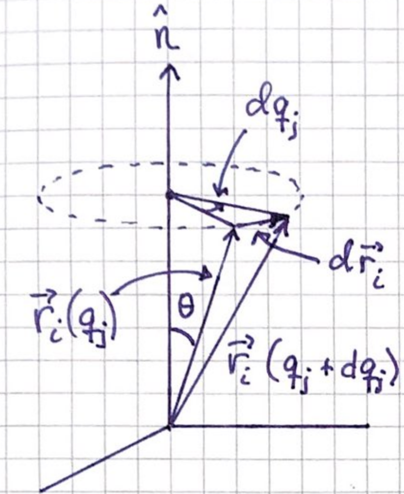
$\Rightarrow$ Here, $\dot{p}_j = Q_j$ is eqn. of motion for total linear momentum.

If q_j is cyclic, i.e., $\frac{\partial L}{\partial q_j} = -\frac{\partial V}{\partial q_j} = Q_j = 0$

$\Rightarrow \dot{p}_j = 0$; conservation law for linear momentum

Rotation:

Consider gen. coord. q_j such that dq_j implies rotation of the whole system around an axis $\hat{n}$.
Again, $\partial T/\partial q_j = 0$ (cf. Ex 1b on p. 11-12), and we will again look closer at $\dot{p}_j = Q_j$ with $p_j = \partial T/\partial \dot{q}_j$ and $Q_j = \sum_i \vec{F}_i \cdot \partial \vec{r}_i/\partial q_j$



$$|d\vec{r}_i| = r_i \sin\theta \, dq_j$$

$$\Rightarrow \left| \frac{\partial \vec{r}_i}{\partial q_j} \right| = r_i \sin\theta$$

$d\vec{r}_i \perp \vec{r}_i$ and $d\vec{r}_i \perp \hat{n}$

$$\Rightarrow \frac{\partial \vec{r}_i}{\partial q_j} = \hat{n} \times \vec{r}_i$$

We will need: $\vec{a} \cdot (\vec{b} \times \vec{c}) = a_i \varepsilon_{ijk} b_j c_k$
$= b_j \varepsilon_{jki} c_k a_i$
$= \vec{b} \cdot (\vec{c} \times \vec{a})$

$$\Rightarrow p_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \sum_i \hat{n} \cdot (\vec{r}_i \times m_i \vec{v}_i) = \hat{n} \cdot \sum_i \vec{L}_i = \hat{n} \cdot \vec{L}$$

$=$ comp. of angular mom. $\vec{L}$ along $\hat{n}$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \vec{F}_i \cdot (\hat{n} \times \vec{r}_i) = \hat{n} \cdot \sum_i \vec{r}_i \times \vec{F}_i$$

$$= \hat{n} \cdot \sum_i \vec{\tau}_i = \hat{n} \cdot \vec{\tau}$$

$=$ comp. of total torque $\vec{\tau}$ along $\hat{n}$

So, now $\dot{p}_j = Q_j$ is eqn. of motion for total angular momentum.

If q_j is cyclic, $\partial L/\partial q_j = -\frac{\partial V}{\partial q_j} = Q_j = 0$

$\Rightarrow \dot{p}_j = 0$; conservation law for angular momentum

In summary: Translational invariance of the system yields conservation of linear momentum; rotational invariance yields conservation of angular momentum.

2.7 Conservation of energy

Assume $L = L(q_i, \dot{q}_i, t)$ with $V = V(q_i)$. Consider $\frac{dL}{dt}$:

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial L}{\partial t}$$

In 1. sum, Lagrange's eqns. gives $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}$

$\Rightarrow$ With product rule, the two sums are $\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i\right)$

$$\Rightarrow \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) + \frac{\partial L}{\partial t} = 0$$

The energy function: $H(q, \dot{q}, t) = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$

Then, if $\partial L/\partial t = 0$: $dH/dt = 0$

Let us show that (usually!) H equals the total energy, T+V. Assume that T contains only quadratic terms in the generalized velocities, i.e., terms prop. to $\dot{q}_i^2$ or $\dot{q}_i\dot{q}_j$; this is usually the case. Then, T is said to be a homogeneous function of the 2. degree, and

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

$$\Rightarrow \quad H = 2T - L = 2T - (T - V) = T + V$$

$\Rightarrow$ If $\frac{\partial L}{\partial t} = 0$, the total energy is conserved