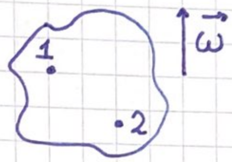
5. Rigid body equations of motion

5.1 Angular momentum and kinetic energy

Rotating rigid body:



$$\vec{\omega}_1 = \vec{\omega}_2 = \vec{\omega}$$

Assume rotation around a fixed point = origo = CM.

Total angular momentum (around the fixed origin):

$$\vec{L} = m_i (\vec{r}_i \times \vec{v}_i) \qquad [\text{Sum convention}]$$

Rotation only $\Rightarrow \vec{v}_i = \vec{\omega} \times \vec{r}_i$

Check this, using ε_{ijk}

$$\Rightarrow \vec{L} = m\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = m_i \left[\vec{\omega}\, r_i^2 - \vec{r}_i\, (\vec{r}_i \cdot \vec{\omega})\right]$$

Components of $\vec{L}$:

$$L_x = m_i \left[\omega_x r_i^2 - x_i (x_i \omega_x + y_i \omega_y + z_i \omega_z)\right]$$

$$= m_i \left[\omega_x (r_i^2 - x_i^2) - \omega_y x_i y_i - \omega_z x_i z_i\right]$$

On the general form (with j and k denoting components)

$$L_j = I_{jk}\, \omega_k \;; \qquad I_{jk} = \text{(moment of) inertia tensor}$$

with elements $I_{xx} = m_i (r_i^2 - x_i^2)$, $I_{xy} = -m_i x_i y_i$ etc.

Continuous mass distribution: $m_i \to dm = \rho\, dV$; $\sum_i \to \int_V$

$$\Rightarrow I_{xx} = \int_V \rho(\vec{r})(r^2 - x^2)\, dV, \qquad I_{xy} = -\int_V \rho(\vec{r})\, xy\, dV$$

Compact: $$I_{jk} = \int_V \rho(\vec{r})\, (r^2 \delta_{jk} - x_j x_k)\, dV \qquad [xyz \to x_1 x_2 x_3]$$

$$\vec{L} = \overleftrightarrow{I}\, \vec{\omega}$$

Clearly $\overleftrightarrow{I}$ is symmetric, $I_{jk} = I_{kj}$, with only real elements, i.e., $\overleftrightarrow{I}$ is hermitean (or self-adjoint, cf. QM). Then $\overleftrightarrow{I}$ can always be diagonalized, i.e., the body set of coords. can be transformed into a coord. system where $\overleftrightarrow{I}$ is diagonal:

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = \text{principal moment of inertia tensor}$$

x_1, x_2, x_3 = principal axes

$L_j = I_j \omega_j$ = angular momentum components $(j=1,2,3)$
(No sum..!)

Ex: Symmetrical top (snurrebass): $I_1 = I_2 \neq I_3$
Spherical top : $I_1 = I_2 = I_3 = I \Rightarrow \vec{L} = I\vec{\omega}$

Kinetic energy for rotation around fixed point:

$$T = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i = \frac{1}{2} m_i \vec{v}_i \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} m_i \vec{\omega} \cdot (\vec{r}_i \times \vec{v}_i) \qquad \text{[Check yourself]}$$

$$= \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega} \overleftrightarrow{I} \vec{\omega} = \frac{1}{2} \omega_j I_{jk} \omega_k$$

If rotation around axis $\hat{n}$, i.e., $\vec{\omega} = \omega \hat{n}$:

$$T = \frac{1}{2} \omega n_j I_{jk} \omega n_k = \frac{1}{2} (n_j I_{jk} n_k) \omega^2 = \frac{1}{2} I \omega^2$$

with $I \equiv n_j I_{jk} n_k = \hat{n} \overleftrightarrow{I} \hat{n}$ = mom. of inertia around rot. axis $\hat{n}$

In coord. system with principal axes x_1, x_2, x_3:

$$T = \frac{1}{2} I_j \omega_j^2$$

5.5 Euler equations of motion

From p.4 : $\vec{\tau} = \dot{\vec{L}}_s$ (measured in the inertial frame)

Measured in the rotating system (p.64; drop index b):

$$\underline{\vec{\tau} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}} \quad \text{(Euler eqns. with one point fixed)}$$

Assume body axes along the principal axes such that $\overleftrightarrow{I}$ is diagonal and $L_j = I_j \omega_j$. Then:

$$\tau_i = \dot{L}_i + \varepsilon_{ijk}\, \omega_j L_k \quad ; \quad i = 1, 2, 3$$

Explicit:

$$\begin{aligned}
\tau_1 &= I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3) \\
\tau_2 &= I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1) \\
\tau_3 &= I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2)
\end{aligned}$$

5.6 Free rotation. Precession

Assume $\vec{\tau} = 0$ and symmetrical body with $I_1 = I_2 \neq I_3$.

$$\begin{aligned}
\Rightarrow I_1 \dot{\omega}_1 &= \omega_2 \omega_3 (I_1 - I_3) \\
I_2 \dot{\omega}_2 &= \omega_1 \omega_3 (I_3 - I_1) \\
I_3 \dot{\omega}_3 &= 0
\end{aligned}$$

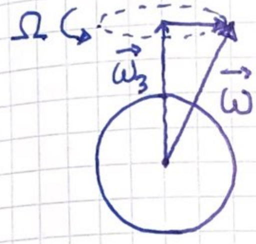
$\Rightarrow \omega_3 = \text{const.}$ (determined by initial conditions)

$$\Rightarrow \dot{\omega}_1 = -\Omega\, \omega_2 \,, \quad \dot{\omega}_2 = \Omega\, \omega_1 \quad ; \quad \Omega = \frac{I_3 - I_1}{I_1}\, \omega_3$$

$\Rightarrow \ddot{\omega}_1 + \Omega^2 \omega_1 = 0$ (after eliminating ω_2)

$$\Rightarrow \omega_1(t) = A \cos \Omega t \,, \quad \omega_2(t) = A \sin \Omega t$$

$\Rightarrow \vec{\omega}$ precesses around the body x_3 axis,
with angular frequency Ω and constant $\omega = |\vec{\omega}|$:



With $\vec{\tau} = 0$ both T and L^2 are conserved:

$$\left.\begin{aligned} T &= \tfrac{1}{2} I_1 A^2 + \tfrac{1}{2} I_3 \omega_3^2 \\ L^2 &= I_1^2 A^2 + I_3^2 \omega_3^2 \end{aligned}\right\} \Rightarrow A(T, L) \text{ and } \omega_3(T, L)$$

Rotating (spinning) planets: Chandler wobble)

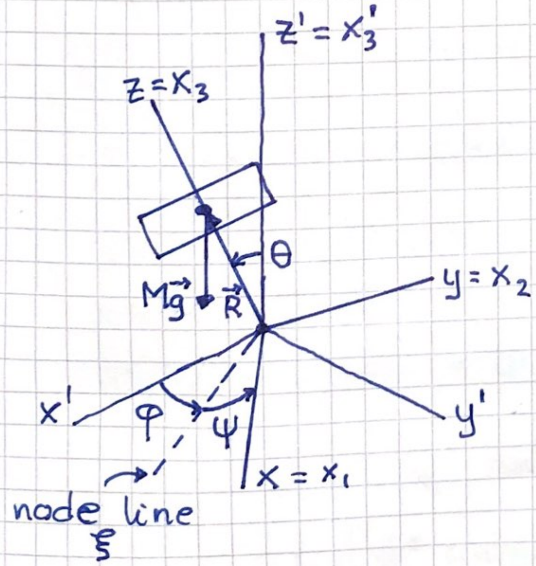
$I_3 - I_1 \ll I_1 \quad \Rightarrow \quad \Omega \ll \omega_3$

Earth: $2\pi/\omega_3 = 1$ day ; $2\pi/\Omega = 433$ days

Mars: $2\pi/\omega_3 \cong$ 1 day 40 min ; $2\pi/\Omega = 207$ days

[A.S. Konopliv et al, Geophysical Research Letters, Oct. 2020]

5.7 Symmetrical top with one point fixed



The top spins around the x_3 axis (a principal axis) with the origin a fixed point.

External torque around origin:

$$\vec{\tau} = \vec{R} \times M\vec{g}$$

$$\left.\begin{aligned} &x_j \text{ fixed in the top} \\ &x_j' \text{ fixed in space} \end{aligned}\right\} \text{NB: } \underline{\text{Opposite}} \text{ in chapter 4 !!}$$

$\Rightarrow \vec{g} = -g\,\hat{x}_3' \; ; \quad \vec{R} = \ell\,\hat{z} = \ell\,\hat{x}_3$

$(\ell = |\vec{R}|)$

Symmetry $\Rightarrow I_1 = I_2 \neq I_3$

Kinetic energy: $T = \frac{1}{2} I_j \omega_j^2 = \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2$

1 point fixed $\Rightarrow$ 3 independent degrees of freedom
Will use the Euler angles φ, θ, ψ to construct the Lagrangian $L = T - V$ $\Rightarrow$ Need $\omega_j(q_i, \dot{q}_i)$, i.e., expressions for the components of $\vec{\omega}$ along the body axes x_j $(j = 1,2,3)$. We transform from space to body by considering $\vec{\omega}'$ as 3 successive rotations with angular velocities $\omega'_\varphi = \dot{\varphi}$, $\omega'_\theta = \dot{\theta}$ and $\omega'_\psi = \dot{\psi}$.

$$\vec{\omega}'_\varphi = \dot{\varphi}\,\hat{x}'_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \quad \Rightarrow \quad \vec{\omega}_\varphi = \mathbb{A}\,\vec{\omega}'_\varphi = \mathbb{B}\mathbb{C}\mathbb{D}\,\vec{\omega}'_\varphi$$

With $\mathbb{A}$ from p. 61:

$$(\omega_\varphi)_1 = \dot{\varphi}\sin\theta\sin\psi, \quad (\omega_\varphi)_2 = \dot{\varphi}\sin\theta\cos\psi, \quad (\omega_\varphi)_3 = \dot{\varphi}\cos\theta$$

$$\vec{\omega}'_\theta = \dot{\theta}\,\hat{\xi} \rightarrow \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{\omega}_\theta = \mathbb{B}\,\vec{\omega}'_\theta$$

With $\mathbb{B}$ from p. 60:

$$(\omega_\theta)_1 = \dot{\theta}\cos\psi, \quad (\omega_\theta)_2 = -\dot{\theta}\sin\psi, \quad (\omega_\theta)_3 = 0$$

$$\vec{\omega}'_\psi = \dot{\psi}\,\hat{x}_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} \quad \Rightarrow \quad \vec{\omega}_\psi = \vec{\omega}'_\psi$$

$$(\omega_\psi)_1 = 0, \quad (\omega_\psi)_2 = 0, \quad (\omega_\psi)_3 = \dot{\psi}$$

We add contributions to each component ω_j:

$$\omega_1 = (\omega_\varphi)_1 + (\omega_\theta)_1 + (\omega_\psi)_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi}$$

$$\Rightarrow \omega_1^2 + \omega_2^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2$$

$$\Rightarrow T = \frac{1}{2} I_1 \left(\dot{\varphi}^2\sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2} I_3 \left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2$$

Pot. energy: $V = Mgl\cos\theta$ $\quad$ [$V = 0$ in $x'y'$ plane]

Lagrangian: $L = T - V = L(\theta, \dot\varphi, \dot\theta, \dot\psi)$

$\Rightarrow \varphi, \psi$ cyclic coords. $\Rightarrow$ p_φ, p_ψ conserved:

$$\left.\begin{aligned} p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\varphi\cos\theta) = I_3\omega_3 \equiv I_1 a \\ p_\varphi &= \frac{\partial L}{\partial \dot\varphi} = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot\varphi + I_3\dot\psi\cos\theta \equiv I_1 b \end{aligned}\right\}$$

These define a and b

Conservative system $\Rightarrow$ energy is conserved:

$$E = T + V = \tfrac{1}{2}I_1(\dot\theta^2 + \dot\varphi^2\sin^2\theta) + \underbrace{\tfrac{1}{2}I_3\omega_3^2}_{\text{const.}} + Mgl\cos\theta$$

From p_ψ: $I_3\dot\psi = I_1 a - I_3\dot\varphi\cos\theta$

Insert this in expression for p_φ:

$I_1\dot\varphi\sin^2\theta + I_1 a\cos\theta = I_1 b$ [I_3 terms cancel]

$$\Rightarrow \dot\varphi = \frac{b - a\cos\theta}{\sin^2\theta} \quad ; \quad \dot\psi = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cdot\cos\theta$$

$$\Rightarrow E' \equiv E - \frac{I_3}{2}\omega_3^2 = \frac{I_1}{2}\dot\theta^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

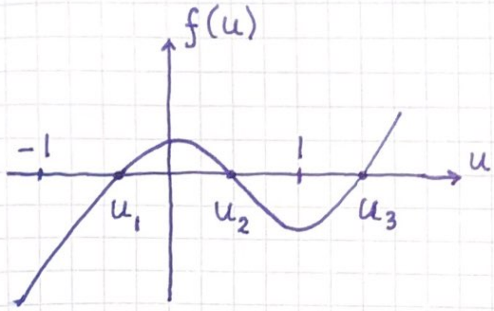
A one-dimensional problem in the variable θ with an effective potential

$$V'(\theta) = Mgl\cos\theta + \tfrac{1}{2}I_1\left(\frac{b - a\cos\theta}{\sin\theta}\right)^2$$

The substitutions $u = \cos\theta$, $\alpha = 2E'/I_1$ and $\beta = 2Mgl/I_1$ and some algebra yields the equation of motion

$$\dot u^2 = \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + (\alpha - b^2) \equiv f(u) \geq 0$$

Solution of $\dot u^2 = f(u)$ yields $\theta(t)$, the <u>nutation</u> of the x_3 axis relative to the vertical space axis x_3'. Next, $\theta(t)$ determines $\dot\varphi(t)$, the <u>precession</u> of x_3 around x_3', and $\dot\psi(t)$, the <u>spinning</u> around x_3.



Solutions of $f(u)=0$ corresponds to $\dot{u}=0$ or $\dot{\theta} = -\dot{u}/\sin\theta = 0$, i.e., values of θ where $\dot{\theta}$ changes sign.

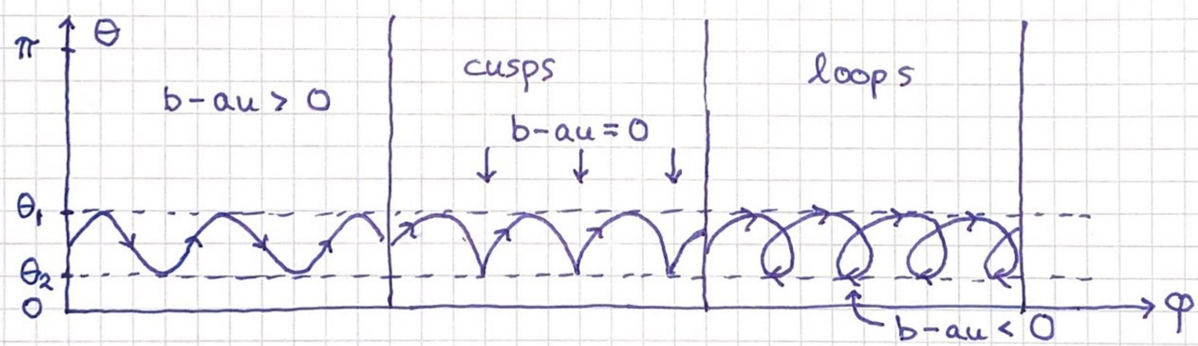
$$\left.\begin{aligned} f(\pm 1) &= -(b \mp a)^2 < 0 \\ f(\pm\infty) &= \pm\infty \end{aligned}\right\}$$ always a solution $u_3 > 1$ which is not physical

Acceptable values of $u = \cos\theta$: $\quad u_1 < u < u_2$

Precession: $\dot{\varphi} = \dfrac{b - a\cos\theta}{\sin^2\theta} = \dfrac{b - au}{1-u^2}$

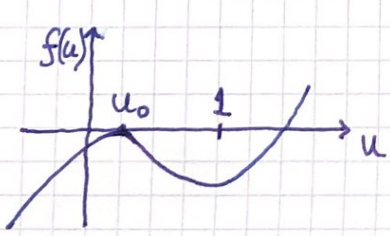
Usually $\dot{\psi} \gg \dot{\varphi}$ for a spinning top.

May have nutation, $\theta_2 < \theta < \theta_1$, with $\dot{\varphi} > 0$ always, $\dot{\varphi} \geq 0$ (cusps) or $\dot{\varphi} \gtrless 0$ (loops):



Regular precession: $\theta(t) = \theta_0 = \text{const.}$

$\Rightarrow f(u) = 0$ for $u = u_1 = u_2 = u_0 = \cos\theta_0$



$f = 0$ and $df/du = 0$ in u_0 (starting point for a deeper analysis of regular prec.)