TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 1

Question 1

L' and L are equivalent provided

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} - \frac{\partial}{\partial q}\frac{dF}{dt} = 0 \qquad (1)$$

We use

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\,\dot{q}$$

in (1) and obtain

$$\begin{aligned} & \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\,\dot{q}\right) - \frac{\partial}{\partial q}\left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\,\dot{q}\right) \\ = \;& \frac{d}{dt}\frac{\partial F}{\partial q} - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\,\dot{q} \\ = \;& \left(\frac{\partial^2 F}{\partial q \partial t} + \frac{\partial^2 F}{\partial q^2}\,\dot{q}\right) - \frac{\partial^2 F}{\partial q \partial t} - \frac{\partial^2 F}{\partial q^2}\,\dot{q} \\ = \;& 0. \end{aligned}$$

In other words, L' and L are equivalent.

Question 2

(a) $x = \ell \sin\beta$, $y = -\ell\cos\beta$

(b) Potential: $V = mgy = -mg\ell\cos\beta$. Kinetic energy: $T = (m/2)(\dot{x}^2 + \dot{y}^2)$. Here, $\dot{x} = \ell\dot{\beta}\cos\beta$ and $\dot{y} = \ell\dot{\beta}\sin\beta$, hence $T = (m/2)(\ell\dot{\beta})^2$. Lagrangian: $L = T - V = (m/2)(\ell\dot{\beta})^2 + mg\ell\cos\beta$.

(c) The Lagrange equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} - \frac{\partial L}{\partial \beta} = 0.$$

We have:

$$\begin{aligned} \frac{\partial L}{\partial \beta} &= -mg\ell\sin\beta \\ \frac{\partial L}{\partial \dot{\beta}} &= m\ell^2\dot{\beta} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}} &= m\ell^2\ddot{\beta} \end{aligned}$$

Thus,

$$m\ell^2\ddot{\beta} + mg\ell\sin\beta = 0$$

or

$$\ddot{\beta} + \frac{g}{\ell}\sin\beta = 0.$$

The same as derived with Newton's 2. law (N2), of course.

Question 3

(a) We see from the figure:

$$\begin{aligned} x_1 &= \ell_1 \sin\beta_1 \\ y_1 &= -\ell_1 \cos\beta_1 \\ x_2 &= \ell_1 \sin\beta_1 + \ell_2 \sin\beta_2 \\ y_2 &= -\ell_1 \cos\beta_1 - \ell_2 \cos\beta_2 \end{aligned}$$

For m_1:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2} m_1 \ell_1^2 \dot{\beta}_1^2 \\ V_1 &= -m_1 g \ell_1 \cos\beta_1 \end{aligned}$$

For m_2:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_2 \left[\ell_1^2 \dot{\beta}_1^2 + \ell_2^2 \dot{\beta}_2^2 + 2\ell_1 \ell_2 \cos(\beta_1 - \beta_2) \dot{\beta}_1 \dot{\beta}_2 \right] \\ V_2 &= -m_2 g \left(\ell_1 \cos\beta_1 + \ell_2 \cos\beta_2 \right) \end{aligned}$$

Here, we used $\cos\beta_1 \cos\beta_2 + \sin\beta_1 \sin\beta_2 = \cos(\beta_1 - \beta_2)$. The Lagrangian:

$$\begin{aligned} L &= T_1 + T_2 - V_1 - V_2 \\ &= \frac{1}{2}(m_1 + m_2)\ell_1^2 \dot{\beta}_1^2 + \frac{1}{2} m_2 \ell_2^2 \dot{\beta}_2^2 + m_2 \ell_1 \ell_2 \dot{\beta}_1 \dot{\beta}_2 \cos(\beta_1 - \beta_2) \\ &+ (m_1 + m_2) g \ell_1 \cos\beta_1 + m_2 g \ell_2 \cos\beta_2 \end{aligned}$$

(b) Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_i} - \frac{\partial L}{\partial \beta_i} = 0 \quad ; \quad i = 1, 2$$

We need to differentiate the expression for L found in task (a). Starting with β_1, we calculate

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_1} &= \frac{d}{dt}\left[(m_1 + m_2) l_1^2 \dot{\beta}_1 + m_2 l_1 l_2 \dot{\beta}_2 \cos(\beta_1 - \beta_2) \right] \\ &= (m_1 + m_2) l_1^2 \ddot{\beta}_1 + m_2 l_1 l_2 \ddot{\beta}_2 \cos(\beta_1 - \beta_2) - m_2 l_1 l_2 \dot{\beta}_2 \left(\dot{\beta}_1 - \dot{\beta}_2 \right) \sin(\beta_1 - \beta_2), \\ \frac{\partial L}{\partial \beta_1} &= -m_2 l_1 l_2 \dot{\beta}_1 \dot{\beta}_2 \sin(\beta_1 - \beta_2) - (m_1 + m_2) g l_1 \sin\beta_1. \end{aligned}$$

The procedure is exactly the same for β_2, but we find slightly different terms:

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_2} &= \frac{d}{dt}\left[m_2 l_2^2 \dot{\beta}_2 + m_2 l_1 l_2 \dot{\beta}_1 \cos(\beta_1 - \beta_2) \right] \\ &= m_2 l_2^2 \ddot{\beta}_2 + m_2 l_1 l_2 \ddot{\beta}_1 \cos(\beta_1 - \beta_2) - m_2 l_1 l_2 \dot{\beta}_1 \left(\dot{\beta}_1 - \dot{\beta}_2 \right) \sin(\beta_1 - \beta_2), \\ \frac{\partial L}{\partial \beta_2} &= m_2 l_1 l_2 \dot{\beta}_1 \dot{\beta}_2 \sin(\beta_1 - \beta_2) - m_2 g l_2 \sin\beta_2. \end{aligned}$$

Terms with $\dot{\beta}_1 \dot{\beta}_2$ cancel, and the resulting coupled equations of motion are:

$$\begin{aligned} (m_1 + m_2)\ell_1^2 \ddot{\beta}_1 + m_2 \ell_1 \ell_2 \ddot{\beta}_2 \cos(\beta_1 - \beta_2) + m_2 \ell_1 \ell_2 \dot{\beta}_2^2 \sin(\beta_1 - \beta_2) + (m_1 + m_2) g \ell_1 \sin\beta_1 &= 0 \\ m_2 \ell_2^2 \ddot{\beta}_2 + m_2 \ell_1 \ell_2 \ddot{\beta}_1 \cos(\beta_1 - \beta_2) - m_2 \ell_1 \ell_2 \dot{\beta}_1^2 \sin(\beta_1 - \beta_2) + m_2 g \ell_2 \sin\beta_2 &= 0 \end{aligned}$$

In the second equation, a common factor $m_2 \ell_2$ may be cancelled.