TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 10

Question 1

The Lorentz transformations are

$$x'_\mu = L_{\mu\nu} x_\nu, \quad x''_\mu = L'_{\mu\nu} x'_\nu.$$

The matrix elements of $\boldsymbol{L}$, $\boldsymbol{L}'$ and $\boldsymbol{L}''$ are

$$L_{11} = L_{22} = 1, \quad L_{33} = L_{44} = \gamma, \quad L_{34} = i\beta\gamma, \quad L_{43} = -i\beta\gamma,$$

$$L'_{11} = L'_{22} = 1, \quad L'_{33} = L'_{44} = \gamma', \quad L'_{34} = i\beta'\gamma', \quad L'_{43} = -i\beta'\gamma'$$

and

$$L''_{11} = L''_{22} = 1, \quad L''_{33} = L''_{44} = \gamma'', \quad L''_{34} = i\beta''\gamma'', \quad L''_{43} = -i\beta''\gamma''$$

respectively. In addition, $\boldsymbol{L}'' = \boldsymbol{L}'\boldsymbol{L}$. By comparing e.g. the 33 and the 34 elements, we find $\beta''\gamma'' = \gamma\gamma'(\beta' + \beta)$, $\gamma'' = \gamma\gamma'(1 + \beta\beta')$. This gives

$$\beta'' = \frac{\beta''\gamma''}{\gamma''} = \frac{\beta + \beta'}{1 + \beta\beta'} \Rightarrow v'' = \frac{v + v'}{1 + vv'/c^2}.$$

This is Einstein's addition formula. Notice that $v < c$, $v' < c \Rightarrow v'' < c$. For $vv' \ll c^2$, we get the regular Galilean velocity addition $v'' = v + v'$.

Question 2

a) In the S frame the tube lights up at the point z at time t. Inserting this into the equations for the Lorentz transformation, we get the coordinates in the system S':

$$\begin{aligned} z' &= \gamma(z - vt) \\ t' &= \gamma\left(t - \frac{vz}{c^2}\right) \end{aligned}$$

In the S frame, the second spacetime event is the tube lighting up at the point $z + \Delta z$ at time t. Again, applying the Lorentz transformations, this event, as seen from the S' system, is given by

$$\begin{aligned} z' + \Delta z' &= \gamma(z + \Delta z - vt) \\ t' + \Delta t' &= \gamma\left(t - \frac{v(z + \Delta z)}{c^2}\right). \end{aligned}$$

b) Using this, we can find the difference in space and time of these two events in the S' system:

$$\begin{aligned} \Delta z' &= \gamma \Delta z \\ \Delta t' &= -\gamma \frac{v\Delta z}{c^2}. \end{aligned}$$

The apparent speed of the lighting up of the tube, in the S' system, is then

$$u = \frac{\Delta z'}{\Delta t'} = \frac{\gamma\Delta z}{-\gamma(v\Delta z/c^2)} = -\frac{c^2}{v}.$$

Question 3

a) As the wave train has length $L = c\Delta t - v\Delta t$ in the S frame, and consists of n wavelengths, it has a wavelength of

$$\lambda = \frac{c\Delta t - v\Delta t}{n}.$$

The frequency of a wave is given by $f = v/\lambda$, and since this is light, $v = c$. The frequency is therefore

$$f = \frac{c}{\lambda} = \frac{cn}{c\Delta t - v\Delta t}.$$

b) The source and the receiver agree on the number of wavelengths in the wave train. In the reference frame of the source S', the frequency and proper time elapsed are related by $n = f_0\Delta t'$. The relation between the proper time of the source and the time according to the receiver in S is given by the time dilation formula, $\Delta\tau = \Delta t' = \Delta t/\gamma$. Putting this all together, we can relate the two frequencies:

$$\begin{aligned} f &= \frac{cf_0\Delta t'}{(c-v)\Delta t} = \frac{1}{\gamma(1-v/c)} f_0 \\ &= \frac{\sqrt{1-(v/c)^2}}{1-v/c} f_0 = \frac{\sqrt{1-v/c}\sqrt{1+v/c}}{\sqrt{1-v/c}\sqrt{1-v/c}} f_0. \end{aligned}$$

With $\beta = v/c$, we get the desired relation

$$f = \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} f_0.$$

c) The only difference from our earlier analysis is that $L = c\Delta t + v\Delta t$, leading us to the relation

$$f = \frac{\sqrt{1-(v/c)^2}}{\sqrt{1+v/c}} f_0 = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} f_0.$$

This is part of the reason for the red shift in astronomy, since the Big Bang made all stars move away from us. This is not the only reason; the expansion of the universe has an important contribution too. However, the underlying theory is beyond the scope of this course.