TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 12

Question 1

Angular momentum:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

Consider first $[p_i, L_j]$ and start with $i = j$, e.g., $i = 1$ (see p 147 in notater97.pdf):

$$\begin{aligned} [p_x, L_x] &= [p_x, yp_z - zp_y] \\ &= [p_x, y]p_z + [p_x, p_z]y - [p_x, z]p_y - [p_x, p_y]z \end{aligned}$$

We have

$$[p_i, x_j] = -\delta_{ij} \quad , \quad [p_i, p_j] = 0$$

and therefore

$$[p_x, L_x] = 0$$

Correspondingly:

$$[p_y, L_y] = [p_z, L_z] = 0$$

Next, consider $[p_i, L_j]$ with $i \neq j$, e.g., $i = 1$ og $j = 2$:

$$\begin{aligned} [p_x, L_y] &= [p_x, zp_x - xp_z] \\ &= [p_x, z]p_x + [p_x, p_x]z - [p_x, x]p_z - [p_x, p_z]x \\ &= 0 + 0 - (-1)p_z - 0 \\ &= p_z \end{aligned}$$

Cyclic change of x, y, z then gives

$$[p_y, L_z] = p_x \quad , \quad [p_z, L_y] = p_z$$

Interchanging indices only gives a change of sign, e.g.,

$$[p_z, L_y] = -p_x$$

In total,

$$[p_i, L_j] = \varepsilon_{ijk} p_k$$

With $L_j = \varepsilon_{jkl} x_k p_l$ this is obtained more directly:

$$\begin{aligned} [p_i, L_j] &= [p_i, \varepsilon_{jkl} x_k p_l] \\ &= [p_i, x_k]\varepsilon_{jkl} p_l \\ &= -\delta_{ik}\varepsilon_{jkl} p_l \\ &= -\varepsilon_{jil} p_l \\ &= \varepsilon_{ijl} p_l \end{aligned}$$

And now I guess we may handle $[x_i, L_j]$ the same way:

$$\begin{aligned} [x_i, L_j] &= [x_i, \varepsilon_{jkl} x_k p_l] \\ &= [x_i, p_l]\varepsilon_{jkl} x_k \\ &= \delta_{il}\varepsilon_{jkl} x_k \\ &= \varepsilon_{jki} x_k \\ &= \varepsilon_{ijk} x_k \end{aligned}$$

Question 2

Hand written solution in Norwegian on next page.

1) $u_x = \frac{dx}{dt} = \frac{dx'}{\gamma(dt' + \frac{v}{c^2}dz')} = \frac{u_x'}{\gamma(1+vu_z'/c^2)}$

$u_y = \frac{dy}{dt} = \frac{u_y'}{\gamma(1+vu_z'/c^2)}$

$u_z = \frac{dz}{dt} = \frac{\gamma(dz'+v\,dt')}{\gamma(dt'+\frac{v}{c^2}dz')} = \frac{u_z'+v}{1+vu_z'/c^2}$

Differensierer:

$$du_x = \frac{du_x'}{\gamma(1+vu_z'/c^2)} - \frac{u_x'}{\gamma(1+vu_z'/c^2)^2}\cdot\frac{v}{c^2}du_z' \xrightarrow{(\vec{u}'=0)} \gamma^{-1}du_x'.$$

Tilsvarende for $du_y = \gamma^{-1}du_y'$.

$$du_z = \frac{du_z'}{1+vu_z'/c^2} - \frac{u_z'+v}{(1+\frac{vu_z'}{c^2})^2}\cdot\frac{v}{c^2}du_z' \to du_z' - \frac{v^2}{c^2}du_z' = (1-\beta^2)du_z'$$

Av $dt = \gamma dt'(1+\frac{v}{c^2}u_z')$ fås $dt = \gamma dt'$.

$\Rightarrow$ $a_x = \frac{du_x}{dt} = (1-\beta^2)a_x'$, tilsvarende $a_y = (1-\beta^2)a_y'$

$a_z = \frac{du_z}{dt} = (1-\beta^2)^{3/2}a_z'$

2) 

$\nu^0 = \frac{dN}{dt^0}$, hvor dN er antall bølger emittert i tiden dt^0 i det instantane hvilesystemet S'.

En har $dt^0 = d\tau$ (egentiden til preparatet).

Hvis t er ~~observert~~ observatørens tid vil $dt = \gamma d\tau$.

Frekvensen ν mottatt i sentrum altså

$$\nu = \frac{dN}{dt} = \frac{dN}{\gamma d\tau} = \frac{\nu^0}{\gamma} = \sqrt{1-\beta^2}\,\nu^0, \quad \beta = \frac{u}{c}$$

Transversal Dopplereffekt.