TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 4

Question 1

a) We have $z = r - s$ for the vertical position of mass 2, and $\dot{z} = \dot{r}$ since s is constant. The various energy contributions are:

$$\begin{aligned} T_1 &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ V_1 &= 0 \\ T_2 &= \frac{1}{2}m\dot{r}^2 \\ V_2 &= mgz = mg(r-s) \end{aligned}$$

This gives the Lagrangian

$$L = T_1 + T_2 - V_1 - V_2 = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg(r-s).$$

b) Lagrange equation for θ:

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \\ \Rightarrow \frac{d}{dt}\left(mr^2\dot{\theta}\right) &= 0 \end{aligned}$$

In other words, the angular momentum $\ell = mr^2\dot{\theta}$ is conserved.

Lagrange equatin for r:

$$\begin{aligned} \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - mg \\ \frac{\partial L}{\partial \dot{\theta}} &= 2m\dot{r} \\ \Rightarrow 2m\ddot{r} - mr\dot{\theta}^2 + mg &= 0 \end{aligned}$$

We replace $\dot{\theta} = \ell/mr^2$ and find

$$2m\ddot{r} - \frac{\ell^2}{mr^3} + mg = 0.$$

c) Circular motion if $\ddot{r} = 0$. Then

$$\frac{\ell^2}{mr_0^3} - mg = 0 \quad \Rightarrow \quad r_0 = (\ell^2/m^2g)^{1/3}.$$

Direct argument: Circular motion if the centrifugal force $mr_0\dot{\theta}^2$ balances the weight mg of the mass below the table, i.e., $r_0\dot{\theta}^2 = g$. With $\ell = mr_0^2\dot{\theta}$, the same expression for r_0 is found.

d) The radial equation is, with $r = r_0 + \rho$:

$$2mr_0 \ddot{+} \rho - \frac{\ell^2}{m(r_0+\rho)^3} + mg = 0.$$

Here, $\ddot{r_0} = 0$, and to leading order

$$\frac{1}{(r_0+\rho)^3} = \frac{1}{r_0^3}\left(1 - \frac{3\rho}{r_0}\right).$$

Thus,

$$2m\ddot{\rho} - \frac{\ell^2}{mr_0^3}\left(1 - \frac{3\rho}{r_0}\right) + mg = 0.$$

In this equation, the constant terms cancel since

$$\frac{\ell^2}{mr_0^3} = \frac{\ell^2}{m}\frac{m^2 g}{\ell^2} = mg.$$

The equation for ρ is therefore

$$\ddot{\rho} + \frac{3g}{2r_0}\rho = 0.$$

This is a harmonic oscillator with angular frequency $\omega = \sqrt{3g/2r_0}$.

Question 2

The force on the particle in $x = 0$, $\boldsymbol{F} = -\nabla V$, acts in the negative x direction. Hence, the y component of the momentum is conserved:

$$mv_0 \sin\alpha = mv\sin\beta \quad \Rightarrow \quad \frac{\sin\alpha}{\sin\beta} = \frac{v}{v_0}.$$

The total energy E is also conserved:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + V_0 \quad \Rightarrow \quad \frac{v}{v_0} = \sqrt{1 - \frac{2V_0}{mv_0^2}} = \sqrt{1 - \frac{V_0}{E}} = n.$$

since $E = mv_0^2/2$. Combination of these equations gives

$$\frac{\sin\alpha}{\sin\beta} = \sqrt{1 - \frac{V_0}{E}} = n.$$

Question 3

a) Each mass m_1 can perform circular motion in the plane with radius a and angular velocity $\dot{\theta}$, and circular motion around the z axis with radius $a\sin\theta$ and angular velocity $\dot{\phi} = \Omega$. The mass m_2 can move along the z axis with velocity $\dot{z}$. Since its position relative to A is $z = -2a\cos\theta$, its velocity is $\dot{z} = 2a\dot{\theta}\sin\theta$. The total kinetic energy is therefore

$$T = 2\cdot\left(\frac{1}{2}m_1 a^2\dot{\theta}^2 + \frac{1}{2}m_1 a^2\sin^2\theta\,\Omega^2\right) + \frac{1}{2}m_2\left(2a\dot{\theta}\sin\theta\right)^2 = m_1 a^2(\dot{\theta}^2 + \sin^2\theta\,\Omega^2) + 2m_2 a^2\dot{\theta}^2\sin^2\theta.$$

Potential energy, with $V = 0$ in A:

$$V = V_1 + V_2 = -2m_1 ga\cos\theta - 2m_2 ga\cos\theta.$$

Lagrangian:

$$L = T - V = m_1 a^2(\dot{\theta}^2 + \sin^2\theta\,\Omega^2) + 2m_2 a^2\dot{\theta}^2\sin^2\theta + 2(m_1+m_2)ga\cos\theta.$$

b) We perform the derivatives and establish the Lagrange equation:

$$\begin{aligned}
\frac{\partial L}{\partial \theta} &= m_1 a^2 \Omega^2 \cdot 2\sin\theta\cos\theta + 2m_2 a^2 \dot{\theta}^2 \cdot 2\sin\theta\cos\theta - 2(m_1+m_2)ga\sin\theta \\
&= a^2(m_1\Omega^2 + 2m_2\dot{\theta}^2)\sin 2\theta - 2(m_1+m_2)ga\sin\theta \\
\frac{\partial L}{\partial \dot{\theta}} &= 2a^2\dot{\theta}(m_1 + 2m_2\sin^2\theta) \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 2a^2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 2a^2\dot{\theta}^2 \cdot 4m_2\sin\theta\cos\theta \\
&= 2a^2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 4a^2 m_2\dot{\theta}^2\sin 2\theta
\end{aligned}$$

Lagrange equation:

$$2a^2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 4a^2 m_2\dot{\theta}^2\sin 2\theta - a^2(m_1\Omega^2 + 2m_2\dot{\theta}^2)\sin 2\theta + 2(m_1+m_2)ga\sin\theta = 0.$$

Or with $\omega_0^2 = 2g/a$:

$$2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 4m_2\dot{\theta}^2\sin 2\theta - (m_1\Omega^2 + 2m_2\dot{\theta}^2)\sin 2\theta + \omega_0^2(m_1+m_2)\sin\theta = 0.$$

Rotational equilibrium means rotation around the z axis with constant $\theta = \theta_0$, i.e., $\dot{\theta} = \ddot{\theta} = 0$:

$$-m_1\Omega^2\sin 2\theta_0 + \omega_0^2(m_1+m_2)\sin\theta_0 = 0,$$

or

$$\sin\theta_0\left[\omega_0^2(m_1+m_2) - 2m_1\Omega^2\cos\theta_0\right] = 0.$$

There are two solutions, $\theta_0 = 0$ and

$$\cos\theta_0 = \frac{(m_1+m_2)\omega_0^2}{2m_1\Omega^2}.$$

Clearly, the right hand side of this equation cannot be larger than 1, so a nonzero value of θ_0 is only possible if Ω exceeds the threshold value

$$\Omega_{\min} = \sqrt{\frac{m_1+m_2}{2m_1}}\,\omega_0.$$

c) We set $m_1 = m_2 = m$ and collect terms that contain $\dot{\theta}$ as a kinetic energy and terms without $\dot{\theta}$ as an effective potential for the one-dimensional problem with θ as coordinate:

$$L = ma^2(1 + 2\sin^2\theta)\dot{\theta}^2 - V'(\theta)$$

with

$$V'(\theta) = -ma^2(\Omega^2\sin^2\theta + 2\omega_0^2\cos\theta).$$

We assume $\Omega > \omega_0$. (With equal masses, $\Omega_{\min} = \omega_0$ above.) Zero derivative of V' now corresponds to equilibrium:

$$\frac{dV'}{d\theta} = -ma^2(2\Omega^2\sin\theta\cos\theta - 2\omega_0^2\sin\theta) = 0.$$

Solutions are $\theta = 0$ and $\cos\theta = \omega_0^2/\Omega^2$, in agreement with what we found in b).

Check yourself whether these values of θ correspond to stable or unstable equilibria when $\Omega > \omega_0$. If $\Omega < \omega_0$, we know that $\theta = 0$ is the only equilibrium angle, which must then be stable.