TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 7

Question 1

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - V(r,\theta,z)$$

Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad , \quad L = L(q,\dot{q})$$

which yields

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 &= -\frac{\partial V}{\partial r} \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) + \frac{\partial V}{\partial \theta} &= 0 \\ m\ddot{z} + \frac{\partial V}{\partial z} &= 0 \end{aligned}$$

Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad , \quad H = H(p,q)$$

The Hamiltonian is

$$H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + V(r,\theta,z)$$

Here, we have used $p_i = \partial L/\partial \dot{q}_i$, which gives

$$p_r = m\dot{r} \quad , \quad p_\theta = mr^2\dot{\theta} \quad , \quad p_z = m\dot{z}$$

and next eliminated $\dot{q}_i$ from the expression for T. Hamilton's equations are

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad , \quad \dot{p}_\theta = -\frac{\partial V}{\partial \theta} \quad , \quad \dot{p}_z = -\frac{\partial V}{\partial z}$$

The three equations for $\dot{q}_i$ are already in place, via the expressions for p_i.

Question 2

a) We will show that $x(t)$ and $y(t)$ obey the equations

$$\ddot{x} - 2\Omega\dot{y} + \omega_0^2 x = 0 \tag{1}$$
$$\ddot{y} + 2\Omega\dot{x} + \omega_0^2 y = 0 \tag{2}$$

where $\Omega = \omega\sin\theta$ and $\omega_0^2 = g/l$. The forces acting on the sphere, measured in the Realfagbygg coordinate system, are:

- gravity $m\boldsymbol{g} = -mg\hat{z}$
- the wire $\boldsymbol{S} = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$

- the Coriolis force $\boldsymbol{F}_{\text{cor}} = -2m\boldsymbol{\omega} \times \boldsymbol{v}$

We neglect the centrifugal force $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$.
The angular velocity of earth is

$$\boldsymbol{\omega} = \omega \sin\theta \ \hat{z} + \omega \cos\theta \ \hat{y}$$

The velocity of the sphere is

$$\boldsymbol{v} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

We assume small oscillations, hence we may write

$$S = |\boldsymbol{S}| \simeq mg$$

and further assume that

$$\dot{z} \ll \dot{x}, \dot{y}$$

The cross product in the Coriolis term is then approximately

$$\begin{aligned}
\boldsymbol{\omega} \times \boldsymbol{v} &\simeq (\omega \sin\theta \ \hat{z} + \omega \cos\theta \ \hat{y}) \times (\dot{x}\hat{x} + \dot{y}\hat{y}) \\
&= \dot{x}\omega \sin\theta \hat{y} - \dot{y}\omega \sin\theta \hat{x} - \dot{x}\omega \cos\theta \hat{z} \\
&= \Omega(\dot{x}\hat{y} - \dot{y}\hat{x}) - \dot{x}\omega \cos\theta \hat{z}
\end{aligned}$$

Consider the components of the wire force. When the distance from the origin is

$$\boldsymbol{r} = x\hat{x} + y\hat{y},$$

we have (with an angle β between the z axis and the wire)

$$\begin{aligned}
|S_x| &= S \sin\beta \sin\phi = S\frac{r}{l}\frac{x}{r} \simeq mg\frac{x}{l} \\
|S_y| &= S \sin\beta \cos\phi = S\frac{r}{l}\frac{y}{r} \simeq mg\frac{y}{l} \\
|S_z| &= S \cos\beta = S\frac{l-z}{l} \simeq mg\frac{l-z}{l}
\end{aligned}$$

The horizontal component of $\boldsymbol{S}$ is all the time directed towards the origin, so we have $S_x > 0$ when $x < 0$, the same for S_y, and opposite when $x > 0$, or $y > 0$. Hence,

$$\begin{aligned}
S_x &= -mgx/l = -m\omega_0^2 x \\
S_y &= -mgy/l = -m\omega_0^2 y \\
S_z &= mg(1 - z/l)
\end{aligned}$$

The equations of motion for the sphere are

$$m\boldsymbol{a} = \boldsymbol{F} = m\boldsymbol{g} + \boldsymbol{S} + \boldsymbol{F}_{\text{cor}}$$

For the x component:

$$m\ddot{x} = -m\omega_0^2 x + 2m\Omega\dot{y}$$

For the y component:

$$m\ddot{y} = -m\omega_0^2 y - 2m\Omega\dot{x}$$

Which is what we were supposed to derive, for the movement in the xy plane.

b) With $u = x + iy$:

$$\begin{aligned}
\dot{u} &= \dot{x} + i\dot{y} \\
\ddot{u} &= \ddot{x} + i\ddot{y}
\end{aligned}$$

Multiply equation (2) with i and add this to equation (1). This gives

$$\ddot{x} + i\ddot{y} + 2i\Omega(\dot{x} + i\dot{y}) + \omega_0^2(x + iy) = 0$$

i.e.

$$\ddot{u} + 2i\Omega\dot{u} + \omega_0^2 u = 0$$

c) We try $u \sim \exp(\alpha t)$ as solution. This yields

$$\alpha^2 + 2i\Omega\alpha + \omega_0^2 = 0$$

with solutions

$$\begin{aligned} \alpha &= -i\Omega \pm i\sqrt{\omega_0^2 + \Omega^2} \\ &\simeq -i\Omega \pm i\omega_0 \end{aligned}$$

since $\omega_0 \gg \Omega$. The general solution for u is therefore

$$\begin{aligned} u(t) &= e^{i\Omega t}\left(Ae^{i\omega_0 t} + Be^{-i\omega_0 t}\right) \\ &= Ce^{-i\Omega t}\cos(\omega_0 t + \gamma) \end{aligned}$$

The complex conjugate of u is then

$$u^*(t) = C^* e^{i\Omega t}\cos(\omega_0 t + \gamma)$$

Since $u = x + iy$, and therefore $u^* = x - iy$, we find

$$\begin{aligned} x &= \frac{1}{2}(u + u^*) = \cos(\omega_0 t + \gamma) \cdot \Re\left\{Ce^{-i\Omega t}\right\} \\ y &= \frac{1}{2i}(u - u^*) = \cos(\omega_0 t + \gamma) \cdot \Im\left\{Ce^{-i\Omega t}\right\} \end{aligned}$$

With the given initial conditions $x = y = 0$ at $t = 0$:

$$\begin{aligned} 0 &= \cos\gamma \cdot \Re C \\ 0 &= \cos\gamma \cdot \Im C \end{aligned}$$

Since we cannot have $C = 0$ (in that case $u = 0$), we must have $\cos\gamma = 0$, i.e., $\gamma = \pi/2$, and therefore $\cos(\omega_0 t + \gamma) = \sin\omega_0 t$. Next, we calculate $\dot{x}$ and $\dot{y}$:

$$\begin{aligned} \dot{x} &= \omega_0 \cos\omega_0 t \Re\left\{Ce^{-i\Omega t}\right\} + \sin\omega_0 t \Re\left\{-i\Omega Ce^{-i\Omega t}\right\} \\ \dot{y} &= \omega_0 \cos\omega_0 t \Im\left\{Ce^{-i\Omega t}\right\} + \sin\omega_0 t \Im\left\{-i\Omega Ce^{-i\Omega t}\right\} \end{aligned}$$

Insert $\dot{x} = v_0$ and $\dot{y} = 0$ at $t = 0$:

$$\begin{aligned} v_0 &= \omega_0 \Re C \\ 0 &= \omega_0 \Im C \end{aligned}$$

which gives

$$\begin{aligned} \Im C &= 0 \\ \Re C &= \frac{v_0}{\omega_0} \end{aligned}$$

The complete solution is

$$
\begin{aligned}
x(t) &= \sin\omega_0 t \cdot \Re\left\{\frac{v_0}{\omega_0} e^{-i\Omega t}\right\} = \frac{v_0}{\omega_0}\cos\Omega t \sin\omega_0 t \\
y(t) &= \sin\omega_0 t \cdot \Im\left\{\frac{v_0}{\omega_0} e^{-i\Omega t}\right\} = -\frac{v_0}{\omega_0}\sin\Omega t \sin\omega_0 t
\end{aligned}
$$

which corresponds to a harmonic oscillations in the xy plane, with angular frequency ω_0, where the plane of oscillation rotates clockwise, with period

$$
T = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega \sin\theta} \simeq 26.8 \text{ h}
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