TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 9

Question 1

The inertia tensor $\boldsymbol{I}$ is given by

$$\boldsymbol{L} = \boldsymbol{I}\,\boldsymbol{\omega}$$

where $\boldsymbol{L}$ is the angular momentum and $\boldsymbol{\omega}$ is the angular frequency. With summation over repeated indices, as usual, we have

$$\begin{aligned}
\boldsymbol{L} &= \boldsymbol{r}_i \times \boldsymbol{p}_i \\
&= \boldsymbol{r}_i \times m_i \boldsymbol{v}_i \\
&= m_i \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i)
\end{aligned}$$

and found in the lectures

$$\boldsymbol{L} = m_i \left[r_i^2\, \boldsymbol{\omega} - (\boldsymbol{r}_i \cdot \boldsymbol{\omega}) \boldsymbol{r}_i \right]$$

This yields, for the elements of $\boldsymbol{I}$:

$$\begin{aligned}
I_{xx} &= m_i(r_i^2 - x_i^2) = m_i(y_i^2 + z_i^2) \\
I_{yy} &= m_i(r_i^2 - y_i^2) = m_i(x_i^2 + z_i^2) \\
I_{zz} &= m_i(r_i^2 - z_i^2) = m_i(x_i^2 + y_i^2) \\
I_{xy} &= -m_i x_i y_i = I_{yx} \\
I_{xz} &= -m_i x_i z_i = I_{zx} \\
I_{yz} &= -m_i y_i z_i = I_{zy}
\end{aligned}$$

a) Relative to the axes x, y, z:

$$\begin{aligned}
I_{xx} &= Ma^2 + Ma^2 + ma^2 + ma^2 = 2(m+M)a^2 \\
I_{yy} &= 2(m+M)a^2 \\
I_{zz} &= 4(m+M)a^2 \\
I_{xy} &= I_{yx} = -Ma^2 - M(-a)^2 - ma(-a) - m(-a)a = 2(m-M)a^2 \\
I_{xz} &= I_{zx} = I_{yz} = I_{zy} = 0
\end{aligned}$$

b) Relative to the axes x', y', z':

$$\begin{aligned}
I_{x'x'} &= 2M(\sqrt{2}a)^2 = 4Ma^2 \\
I_{y'y'} &= 2m(\sqrt{2}a)^2 = 4ma^2 \\
I_{z'z'} &= 4(m+M)a^2 \\
I_{x'y'} &= I_{y'x'} = 0 \\
I_{x'z'} &= I_{z'x'} = I_{y'z'} = I_{z'y'} = 0
\end{aligned}$$

i.e., diagonal.

Question 2

Let x_1 and x_2 denote the distance of the two blocks from the wall. The kinetic and potential energy of the system is then

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \quad V = \frac{1}{2}k(x_1^2 + (x_1 - x_2)^2).$$

Writing out the Lagrangian in matrix form gives

$$L = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}k(2x_1^2 - 2x_1x_2 + x_2^2) = \frac{1}{2}\left(T_{ij}\dot{x}_i\dot{x}_j + V_{ij}x_ix_j\right),$$

where

$$T_{ij} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad V_{ij} = k\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

If we introduce $\omega_0^2 = k/m$, the equation for the eigenfrequencies is

$$\left|\boldsymbol{V} - \omega^2\boldsymbol{T}\right| = \begin{vmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0.$$

This gives a second order equation for ω^2, with the solutions

$$\omega^2 = \frac{1}{2}\left(3\omega_0^2 \pm \sqrt{(3\omega_0^2)^2 - 4\omega_0^4}\right) = \frac{3 \pm \sqrt{5}}{2}\omega_0^2.$$

The (relative) amplitudes of the two masses in the two normal modes are determined by inserting the solutions for ω^2, one by one, into the original set of equations. This gives

$$A_{11} = -\left(\frac{1 + \sqrt{5}}{2}\right)A_{21}$$

and

$$A_{12} = \left(\frac{2}{1 + \sqrt{5}}\right)A_{22}.$$

Question 3

We need to solve the set of equations

$$
\begin{aligned}
(k-\omega_\alpha^2 m)A_{1\alpha} - kA_{2\alpha} &= 0 \\
-kA_{1\alpha} + (2k-\omega_\alpha^2 M)A_{2\alpha} - kA_{3\alpha} &= 0 \\
-kA_{2\alpha} + (k-\omega_\alpha^2 m)A_{3\alpha} &= 0
\end{aligned}
$$

Here, $A_{j\alpha}$ corresponds to the amplitude of atom j in normal mode α.

$\omega_1 = 0$ gives $A_{11} = A_{21} = A_{31}$, i.e., translation of the molecule (along the molecule axis).

$\omega_2 = \sqrt{k/m}$ gives $A_{22} = 0$ and $A_{32} = -A_{12}$. The symmetric mode, where the C atom is at rest and the O atoms oscillate with equal amplitude in opposite direction.

$\omega_3 = \sqrt{(2m+M)k/mM}$ gives $A_{13} = A_{33}$ and $A_{23} = -(2m/M)A_{13}$. The antisymmetric mode where the O atoms oscillate with equal amplitude in the same direction and the C atom in the opposite direction, with an amplitude such that the center of mass is at rest.

Question 4

Both a) and b) are derived on pp 111 and 112 in the Norwegian compendium from 1997.