TFY4345 Classical Mechanics. Department of Physics, NTNU.

ASSIGNMENT 6

Question 1

A particle with mass m moves in an attractive potential $V(r)$. Show, on the basis of energy conservation, how the problem can be looked upon as a one-dimensional problem with the effective potential $V_{\text{eff}}(r)$. What is the condition for the particle to reach the scattering center, $r = 0$?

Hint: Once setting the condition between E and V_{eff}, consider the limit $r \to 0$.

Question 2

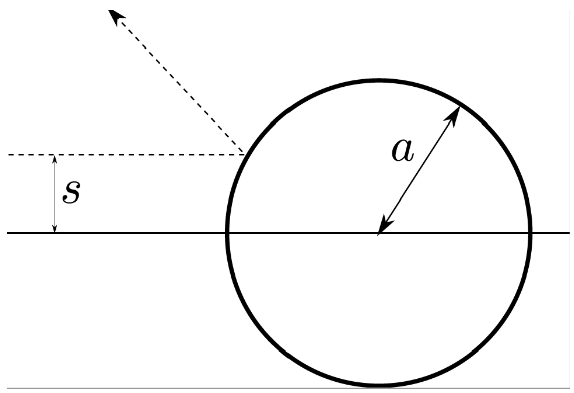

Figure 1: Scattering off a hard sphere.

a) Find the differential cross section $\sigma(\theta)$ for scattering off a hard sphere of radius a.

Hint: When the impact parameter is $s = 0$, the scattering angle is $\theta = \pi$, and when $s > a$, the scattering angle is $\theta = 0$ (no scattering). Use simple geometrical considerations to find a relation between the impact parameter s and the scattering angle θ.

b) Calculate the total cross section σ. Is the result as expected?

Question 3

A hard sphere has radius a. For $r > a$, the sphere provides a Kepler potential $V(r) = -k/r$, with $k > 0$. Particles coming in from infinity have mass m and original velocity v_0. Particles with impact parameter $s \leq s_{\max}$ will hit the sphere's surface. Find $s_{\max}$ and the corresponding "effective" scattering cross section $\sigma_{\text{eff}} = \pi s_{\max}^2$.

Hint: Consider conservation of energy and angular momentum.

Question 4

a) The solution to the Kepler problem (particle in a central force potential $V = -k/r$) is given by the following expression (in polar coordinates):

$$r = \frac{p}{1 + \varepsilon \cos\theta}.$$

Here, p and ε are constants whose definitions can be found in the lecture notes. Use the expression defining p and ε to show that the total energy is

$$E = -\frac{k}{2p}(1 - \varepsilon^2).$$

Use the virial theorem to find the average kinetic energy $\langle T \rangle$, and the average potential energy $\langle V \rangle$.

b) The average potential energy can be calculated as the average over one orbital period:

$$\langle V \rangle = \frac{1}{t_p} \int_0^{t_p} V \, dt,$$

where t_p is the total orbital period (time for one complete orbit). Find $\langle V \rangle$ by direct calculation of this integral (not using the virial theorem). From the lecture notes we have

$$t_p = \frac{2m}{\ell} \pi ab = \frac{2\pi m}{\ell} \frac{p}{1 - \varepsilon^2} \frac{p}{\sqrt{1 - \varepsilon^2}} = \frac{2\pi m}{\ell^2} \frac{1}{(1 - \varepsilon^2)^{3/2}}.$$

Hint 1: Change the variable of integration from t to θ.
Hint 2: The residue method (E. Kreyzig, 9th ed., chapter 16.4) gives

$$\int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \cos\theta} = \frac{2\pi}{\sqrt{1 - \varepsilon^2}}.$$

Hint 3:

$$\int_0^{2\pi} \frac{d\theta \cos\theta}{(1 + \varepsilon \cos\theta)^2} = -\frac{d}{d\varepsilon} \int_0^{2\pi} \frac{d\theta}{1 + \varepsilon \cos\theta}$$

c) The average kinetic energy is

$$\langle T \rangle = \frac{1}{t_p} \int_0^{t_p} T \, dt = \frac{1}{t_p} \int_0^{t_p} dt \, \frac{1}{2} m \left(\frac{d\boldsymbol{r}}{dt} \right)^2.$$

Find $\langle T \rangle$ by direct calculation of this integral.

Hint: Do the integral by parts, and use Newton's second law

$$m \frac{d^2 \boldsymbol{r}}{dt^2} = -\nabla V = -\frac{k}{r^3} \boldsymbol{r}.$$

Note also that the integration (by parts) is taken over a full period where the particle has returned to its initial position.