TFY4345 Classical Mechanics. Department of Physics, NTNU.

SOLUTION ASSIGNMENT 3

Question 1

a) We want to solve the second order differential equation

$$\ddot{\theta} + \omega_0^2\theta - \frac{A\gamma^2}{\ell}\cos(\gamma t) = 0.$$

To rid ourselves of the derivatives, we go to Fourier space. We use the following normalization for the Fourier transform $\theta(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \exp(-i\omega t)\theta(t), \theta(t) = \int_{-\infty}^{\infty} d\omega \exp(i\omega t)\theta(\omega)$. At first, we calculate the Fourier transform $\tilde{f}(\omega)$ of $f(t) = \cos(\gamma t)$. $\tilde{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \exp(-i\omega t)\cos(\gamma t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt \exp(-i\omega t)\left[\frac{\exp(i\gamma t)+\exp(i\gamma t)}{2}\right]$ $= \frac{1}{2}[\delta(\omega-\gamma)+\delta(\omega+\gamma)]$. To arrive at the last equality, we have recognized the Fourier representation of the Dirac delta function

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \exp[-i\omega(x-a)].$$

We now express the differential equation above in Fourier space $0 = \int_{-\infty}^{\infty} d\omega\, \tilde{\theta}(\omega)\left[\frac{d^2}{dt^2} + \omega_0^2\right]\exp(i\omega t) -$ $\frac{A\gamma^2}{2\ell}[\delta(\omega-\gamma)+\delta(\omega+\gamma)]\exp(i\omega t)$
$= \int_{-\infty}^{\infty} d\omega \left\{\tilde{\theta}(\omega)[\omega_0^2 - \omega^2] - \frac{A\gamma^2}{2\ell}[\delta(\omega-\gamma)+\delta(\omega+\gamma)]\right\}\exp(i\omega t)$. This equality certainly holds when the integrand vanishes. Therefore, identify $\tilde{\theta}$ by requiring that the terms in the integrand cancel. Fourier transforming back, we find $\theta(t)$. $\theta(t) = \int_{-\infty}^{\infty} d\omega \exp(i\omega t)\frac{A\gamma^2}{2\ell}\frac{1}{\omega_0^2-\omega^2}[\delta(\omega-\gamma)+\delta(\omega+\gamma)]$
$= \frac{A\gamma^2}{2\ell}\frac{1}{\omega_0^2-\gamma^2}[\exp(i\gamma t)\exp(-i\gamma t)]$
$= \frac{A\gamma^2}{\ell}\frac{1}{\omega_0^2-\gamma^2}\cos(\gamma t)$. Plugging this back into the differential equation, we may check that this is in fact a solution. When $\gamma^2 \to \omega_0^2 = gl$, the solution grows arbitrarily large, at which point the small angle approximation certainly becomes nonsense.
Is this solution unique? Imagine adding to $\theta(t)$ another function $\theta_0(t)$, which solves $\ddot{\theta}_0 + \omega_0^2\theta_0 = 0$. The sum of $\theta(t) + \theta_0(t)$ still solves the differential equation for the driven pendulum, meaning that the solution is not unique.