Department of Physics

Exam TFY4275/FY8907 Stochastic Processes and Transport Theory
Examination date: December 22, 2022
Examination time 09:00-13:00:
Permitted examination support material: C
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<u>Problem 1.</u> (Points: 10+10+10= 30)

Consider the stochastic difference equation

$$z_{n+1} = z_n + \sigma_n$$

where $\{\sigma_n\}$ are statistically independent, identically distributed random numbers whose probability distribution $\mathbb{P}$ is given by

$$\mathbb{P}(\sigma_n = \varepsilon) = p\delta_{\varepsilon,1} + q\delta_{\varepsilon,-1}$$

where δ are Kronecker deltas. p and q are positive real number. The initial value of z_n is denoted z_0.

a) Express $\langle z_n \rangle$ and $\langle z_n^2 \rangle$ in terms of expectation values involving σ_i.

b) Compute $\langle \sigma_i \rangle$ and $\langle \sigma_i \sigma_j \rangle$.

c) Express $\langle z_n^2 \rangle - \langle z_n \rangle^2$ in terms of p and n, and give a physical interpretation of what sort of stochastic process this is. (Eliminate q from the expression by using an appropriate relation between p and q).

<u>Problem 2.</u> (Points: 10+10+10=30)

In the relaxation time approximation, the Boltzmann equation for the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = -\frac{(f - f_0)}{\tau}$$

Consider fermions with charge $-e$ and mass m, and let the system be subject to a weak electric field that varies with time but is constant in space, $\mathbf{E}(t) = \mathbf{E}_0 e^{i\omega t}$. Thus, allow for a time-dependent $f(\mathbf{v}, t)$.

a) Give the general expression for the heat-current and the electrical current for this system, paying special attention to which part of the distribution function that gives a nonzero contribution to the currents.

b) Solve the Boltzmann-equation for this system, for weak electric field, calculating to linear order in $\mathbf{E}(t)$.

c) Use the solution to the Boltzmann-equation to compute the electrical conductivity and the thermoelectric transport coefficient for this system. (The thermoelectric transport coefficient is defined as the coefficient that gives the heat-current as a function of electric field).

<u>Problem 3.</u> (Points: 10+10+10=30)

The Fokker-Planck equation is given by

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left(\alpha_1(y)P(y,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\alpha_2(y)P(y,t)\right)$$

a) Derive the Fokker-Planck equation from the general Master-equation for a continuous stochastic variable, stating clearly under what limitations it is derived.

b) Give an interpretation of the various quantities entering into this equation, and what the origin of $\alpha_{1,2}$ is.

c) State what this equation describes, and give a physical interpretation of the two terms on the right hand side of the equation.

Formulae that may be useful (it is presumed that the candidates will be able to interpret the symbols entering the formulea):

The equilibrium Fermi distribution function:

$$f_0(x) = \frac{1}{e^{\beta x} + 1}$$

Poisson distribution

$$P_n(\lambda) = \frac{\lambda^n}{n!}\, e^{-\lambda}$$

Master-equation for a continuous stochastic variable y:

$$\frac{\partial P(y,t)}{\partial t} = \int d\eta \left[\omega(y-\eta;\eta)P(y-\eta,t) - \omega(y;\eta)P(y,t)\right]$$

Here, $\omega(y;\eta)$ is a transition *rate*, from the state y to a state $y+\eta$.

Master-equation for a discrete stochastic variable n:

$$\frac{dP_n(t)}{dt} = \sum_{n'} \left[\omega_{n,n'} P_{n'}(t) - \omega_{n'n} P_n(t)\right]$$

Here, $\omega_{n,n'}$ is a transition *rate*, from the state n' to a state n.