Problem 1. (Points: 10+10+10+10+10= 50)

The discrete Master equation governing a stochastic process for a discrete random variable n is given by

$$\frac{dP_n(t)}{dt} = \sum_{n'} \left[\omega_{n,n'} \ P_{n'}(t) - \omega_{n',n} P_n(t)\right].$$

a) Explain what the various terms in the equation mean, and under what conditions the equation is derived.

b) Consider a stochastic process where $\omega_{n,n'}$ is given by

$$\omega_{n,n'} = \lambda(n')\delta_{n',n+1}(1 - 2\delta_{n,n'-2}) + \kappa(n')\delta_{n',n-1}(1 + 2\delta_{n,n'+2}),$$

where $\delta_{i,j}$ is the Kronecker symbol, $n \in \mathbb{Z}$ (the set of all integers), and $P_n(0) = \delta_{n,0}$. Perform the summation over n' in the Master-equation and give the resulting equation for $P_n(t)$.

c) Consider now the case where $\lambda(n) = \rho$ and $\kappa(n) = \eta$, where ρ, η are independent of n. Introduce the generating functional $F(z,t) \equiv \sum_n z^n P_n(t)$ and show that the equation for $F(z,t)$ is given by

$$z\frac{\partial F}{\partial t} = \left[\rho(1-z) + \eta(z-1)z\right] F(z,t).$$

What is the initial condition on $F(z,t)$?

d) Solve the equation for $F(z,t)$ and show that it is given by

$$F(z,t) = e^{-(\rho+\eta)t} \ H(\rho t/z) \ H(\eta z t),$$

thus determining the function $H(x)$.

e) Use the definition of $F(z,t)$ to derive general relations between $\langle n \rangle$ and $\langle n^2 \rangle$ and $F(z,t)$. Use the result for $F(z,t)$ in **d)** to compute $\langle n \rangle$ and $\langle (n - \langle n \rangle)^2 \rangle$ explicitly. What sort of discrete stochastic process does $\omega_{n,n'}$ describe?

<u>Problem 2.</u> (Points: 10+10+10+10+10+10=60)
The Boltzmann-equation for the one-particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$ is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \int d^3 v_1 \int \; d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}| \; [f' f_1' - f f_1]$$

The collision term on the right hand side of the equation describes a collision $(\mathbf{v}, \mathbf{v}_1) \rightarrow (\mathbf{v}', \mathbf{v}_1')$, with $f = f(\mathbf{r}, \mathbf{v}, t)$, $f_1 = f(\mathbf{r}, \mathbf{v}_1, t)$, and correspondingly for (f', f_1'). Finally, $\sigma(\Omega)$ is the differential scattering cross section for the collision.

a) A velocity-average of a quantity $A(\mathbf{v})$ may be obtained from f as

$$\langle A \rangle \equiv \frac{1}{n} \int d^3 v A f$$

where $n = \int d^3 v f$, where n is a number density. Show that we have the following macroscopic equation for any $A(\mathbf{v})$

$$\frac{\partial \left(n \langle A \rangle\right)}{\partial t} + \frac{\partial \left(n \langle A v_j \rangle\right)}{\partial x_j} - a_j n \langle \frac{A}{\partial v_j} \rangle = \int d^3 v \int d^3 v_1 \int \; d\Omega \; \sigma(\Omega) \; |\mathbf{v}_1 - \mathbf{v}| \; [f' f_1' - f f_1] \, A(\mathbf{v})$$

b) Define what is meant by a collision invariant, and show that the right hand side of the above equation vanishes whenever A is a collision invariant.

c) Now let $A = \mathbf{p}$, where $\mathbf{p}$ is the momentum of the particle described by f. Let $\rho = nm$ be the mass-density of the system. Set up the macroscopic equation for $\mathbf{p}$ and give the physical interpretation of it.

d) The hydrodynamic conservation laws for mass and momentum are given by, on their most general form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \\ \frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \mathbf{u}) &= \mathbf{F} - \nabla \cdot \mathbb{P} \end{aligned}$$

where $\mathbb{P}$ is the pressure-tensor of the system, $\mathbf{F}$ is an external force acting on a little fluid element of density ρ, and $\mathbf{u} = \langle \mathbf{v} \rangle$. Express $\mathbb{P}$ in terms of an appropriate velocity-average, as defined above, by comparing the above equations with the result you found in **d**).

e) Consider the system close to equilibrium, so that we may use the following Ansatz for f

$$f(\mathbf{r},\mathbf{v},t) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{m}{2k_B T}(\mathbf{v}-\mathbf{u})^2\right)$$

where k_B is Boltzmann's constant and T is temperature, and $(n, T, \mathbf{u})$ could depend on $(\mathbf{r}, t)$. In general, the pressure tensor $\mathbb{P}_{ij}$ of a fluid may be written on the form (need not be shown!)

$$\mathbb{P}_{ij} = p\ \delta_{ij} - \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right) - \zeta\delta_{ij}\nabla\cdot\mathbf{u}$$

Here, p is hydrostatic pressure, while (η, ζ) are properties (viscosities) of the fluid that describe internal friction under shearing and compression. Calculate $\mathbb{P}$ using $f(\mathbf{r},\mathbf{v},t)$ as given above, and from this find expressions for the hydrostatic pressure p, the shear viscosity η, and bulk-viscosity ζ of the system, by comparing the result of your calculation to the form given above.

f) Consider how the general expression for $\mathbb{P}_{ij}$ transforms under time-reversal. How could you have arrived at the results you found for (η, ζ) in **e)** based on such time-reversal considerations?

Formulae that may be useful (it is presumed that the candidates will be able to interpret the symbols entering the formulae):

The equilibrium Fermi distribution function:

$$f_0(x) = \frac{1}{e^{\beta x} + 1}$$

Poisson distribution

$$P_n(\lambda) = \frac{\lambda^n}{n!}\ e^{-\lambda}$$

Binomial expansion

$$(x+y)^N = \sum_{n=0}^{N} \binom{N}{n}\ x^n\ y^{N-n} = \sum_{n=0}^{N}\ \frac{N!}{n!(N-n)!}\ x^n\ y^{N-n}$$

Master-equation for a continuous stochastic variable y:

$$\frac{\partial P(y,t)}{\partial t} = \int d\eta \left[\omega(y-\eta;\eta)P(y-\eta,t) - \omega(y;\eta)P(y,t)\right]$$

Taylor expansion of the exponential function:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Modified Bessel-function of the first kind of order n:

$$I_n(x) = \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2k}}{k!(k+n)!}$$

Gaussian integrals

$$\begin{aligned}
\int d^3v\ v_i\ v_j\ e^{-\alpha v^2} &= \frac{1}{3}\ \delta_{ij} \int d^3v\ v^2\ e^{-\alpha v^2} \\
\int_0^{\infty} dv\ v^2\ e^{-\alpha v^2} &= \frac{1}{4}\ \sqrt{\frac{\pi}{\alpha}}\ \frac{1}{\alpha} \\
\int_0^{\infty} dv\ v^4\ e^{-\alpha v^2} &= \frac{3}{8}\ \sqrt{\frac{\pi}{\alpha}}\ \frac{1}{\alpha^2}
\end{aligned}$$

Three dimensional volume element in spherical coordinates

$$d^3v = d\phi\ \sin\theta d\theta\ v^2 dv$$