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1a)
$$z_{n+1} = z_n + \sigma_n$$

$$z_1 = z_0 + \sigma_0$$

$$z_2 = z_1 + \sigma_1 = z_0 + \sigma_0 + \sigma_1$$

$$\vdots$$

$$z_n = z_0 + \sigma_0 + \sigma_1 + \cdots \sigma_{n-1}$$
$$= z_0 + \sum_{i=0}^{n-1} \sigma_i$$

$$\underline{\underline{\langle z_n \rangle = z_0 + \sum_{i=0}^{n-1} \langle \sigma_i \rangle}} = \underline{\underline{z_0 + n \langle \sigma_i \rangle}}$$

$$z_n^2 = \left(z_0 + \sum_{i=0}^{n-1} \sigma_i\right)\left(z_0 + \sum_{j=0}^{n-1} \sigma_j\right)$$

$$\langle z_n^2 \rangle = z_0^2 + 2 z_0 \sum_{i=0}^{n-1} \langle \sigma_i \rangle + \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle \sigma_i \sigma_j \rangle$$

σ_i are stipulated to be statist. independent, so $\langle \sigma_i \sigma_j \rangle = \langle \sigma_i \rangle \langle \sigma_j \rangle$ NB: $i \neq j$

When $i = j$: $\underline{\langle \sigma_i^2 \rangle = 1}$

b) $\langle \sigma_i \rangle = \sum_{\sigma_i = \pm 1} \sigma_i P(\sigma_i)$

$= \underline{\underline{p - q}}$

$\underline{\langle \sigma_i \sigma_j \rangle :}$

Consider $i=j$ and $i \neq j$ separately, since $\sigma_i^2 = 1$

$\underline{i=j:}$ $\langle \sigma_i \sigma_j \rangle = \langle \sigma_i^2 \rangle = \langle 1 \rangle = \underline{\underline{1}}$

$\underline{i \neq j:}$ $\langle \sigma_i \sigma_j \rangle = \langle \sigma_i \rangle \langle \sigma_j \rangle$
(since $\{\sigma_i\}$ are statistically independent)

$\langle \sigma_i \rangle = p - q$

$(\sigma_i \sigma_j) = (p-q)^2$, $i \neq j$

$= (p-q)^2 (1 - \delta_{ij})$

$\langle \sigma_i \sigma_j \rangle = \underline{\underline{\delta_{ij} + (p-q)^2 (1 - \delta_{ij})}}$

c)

$$\langle Z_n \rangle = Z_0 + n\langle \sigma_i \rangle$$
$$= \underline{\underline{Z_0 + n(p-q)}}$$

$$\underline{\underline{\langle Z_n \rangle^2 = Z_0^2 + 2nZ_0(p-q) + n^2(p-q)^2}}$$

$$\langle Z_n^2 \rangle = Z_0^2 + 2Z_0\, n(p-q)$$
$$+ \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\delta_{ij} + (1-\delta_{ij})(p-q)^2\right]$$

The last term =

$$n + n(n-1)(p-q)^2$$
$$= n\left(1-(p-q)^2\right) + n^2(p-q)^2$$

$$\langle Z_n^2 \rangle = \langle Z_n \rangle^2 + n\left(1-(p-q)^2\right)$$

$$\langle Z_n^2 \rangle - \langle Z_n^2 \rangle = n\left(1-(p-q)^2\right)$$

Normalization of P: $p+q=1$

$$p-q = 2p-1$$
$$1-(2p-1)^2 = 4p(1-p)$$

$$\langle z_n^2\rangle - \langle z_n\rangle^2 = n \cdot 4p(1-p)$$

The stochastic process is an asymmetric random walk with a „drift-velocity)

$$\langle z_{n+1}\rangle - \langle z_n\rangle = p-q$$

where the stochastic variable z_n spreads out diffusively around the moving average, with diffusion constant

$$D = p(1-p)$$

Problem 3

a) Master-equation

$$\frac{\partial P(y,t)}{\partial t} = \int d\eta \left[w(y-\eta;\eta)\, P(y-\eta,t) - w(y;\eta)\, P(y,t) \right]$$

We note that the first term in square brackets is the same as the second term, with $y \to y-\eta$.

Assume that $w(y-\eta,\eta)$ varies slowly with η, and that it is a function that is „short-ranged" in η: transition rates for large jumps in the stochastic variable y are negligible. Then we may Taylor-expand the first term in y, and when subtracting off the second term, we obtain

$$\frac{\partial P(y,t)}{\partial t} = \int d\eta \sum_{k=1}^{\infty} \frac{(-\eta)^k}{k!} \left(\frac{\partial}{\partial y}\right)^k \left(\omega(y;\eta) P(y,t)\right)$$

This is the Kramers-Moyal expansion, and up to this point, it is a formally exact representation of the Master-equation.

Under the assumptions given above, we may truncate the expansion to second order in derivatives

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left[\left(\int d\eta\, \eta\, \omega(y;\eta)\right) P(y,t)\right] + \frac{1}{2}\left[\left(\int d\eta\, \eta^2 \omega(y;\eta)\right) P(y,t)\right]$$

Define

$$\alpha_l(y) \equiv \int d\eta\, \eta^l\, w(y;\eta)$$

Then

$$\frac{\partial P(y,t)}{\partial t} = -\frac{\partial}{\partial y}\left(\alpha_1(y)\, P(y,t)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\alpha_2(y)\, P(y,t)\right)$$

q.e.d.

In addition to the conditions on $w(y;\eta)$ stated above, we also have to remember that the Master-equation itself is derived assuming that the stochastic process is <u>Markovian</u>.

b) y: Stochastic variable

t: Time

$P(y,t)$ time-dependent probability-distribution (strictly speaking transition-probability distribution) for the stochastic variable y.

$$\alpha_1 \equiv \int d\eta \, \eta \, \omega(y,\eta)$$

„Velocity" in y-space and gives rise to a drift in y-space

$$\alpha_2 \equiv \int d\eta \, \eta^2 \, \omega(y,\eta)$$

„Diffusion" constant in y-space and gives rise to diffusive spread of y around the $\langle y \rangle$.

c) The FP-equation describes the temporal evolution of the transition probability P of a stochastic variable y.

On computing $\langle y \rangle$, $\langle y^2 \rangle$ etc, the first ~~term~~ term gives rise to a drift of $\langle y \rangle$

$$\frac{\partial \langle y \rangle}{\partial t} = \langle \alpha_1(y) \rangle$$

while the second term gives rise to diffusive spread of $\langle y^2 \rangle$ as a function of time

$$\frac{\partial \langle y^2 \rangle}{\partial t} = 2 \langle y \alpha_1(y) \rangle + \langle \alpha_2(y) \rangle$$

Problem 2

a) General expressions for heat-current and electrical current

Heat current $\vec{j}_Q = 2\int \frac{d^3p}{(2\pi\hbar)^3} \vec{v}(\varepsilon_p - \mu) f(\vec{p})$

Electrical current: $\vec{j}_e = -2e \int \frac{d^3p}{(2\pi\hbar)^3} \vec{v} f(\vec{p})$

f is a solution to the Boltzmann equation, which in the relaxation-time approximation takes the form

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \vec{a}\cdot\frac{\partial f}{\partial \vec{v}} = -\frac{1}{\tau}(f - f_0)$$

$\vec{a}$: Acceleration due to some external force

$f_0(\vec{p})$: Equilibrium solution.

$$f_0(\vec{p}) = f_0(\varepsilon_{\vec{p}}) = \frac{1}{e^{\beta(\varepsilon_{\vec{p}}-\mu)}+1}$$

Define $f_1 = f - f_0$

Only the f_1-part will contribute to $\vec{j}_Q, \vec{j}_e$.

<u>b)</u> $\vec{a} = \vec{E}(t) = \vec{E}_0 \, e^{i\omega t}$

$f = f(\vec{p}, t)$

$$\frac{\partial f_1}{\partial t} - e\vec{E}_0 \, e^{i\omega t} \cdot \vec{v} \, \frac{\partial f_0}{\partial \varepsilon_p} = -\frac{f_1}{\tau}$$

$$\frac{\partial f_1}{\partial t} + \frac{f_1}{\tau} = + e\vec{E}_0 \, e^{i\omega t} \cdot \vec{v} \, \frac{\partial f_0}{\partial \varepsilon_p}$$

$$\frac{\partial}{\partial t}\left(e^{\frac{t}{\tau}} f_1\right) = e^{\frac{t}{\tau}} \, e \vec{E}_0 \cdot \vec{v} \, \frac{\partial f_0}{\partial \varepsilon_p} \, e^{i\omega t}$$

$$\int_0^t dt' \frac{d}{dt'}\left(e^{\frac{t'}{\tau}} f_1\right)$$

$$= \int_0^t dt' \, e^{\frac{t'}{\tau} + i\omega t'} \left(e \vec{E}_0 \cdot \vec{v} \frac{\partial f_0}{\partial \varepsilon_p}\right)$$

$$e^{\frac{t}{\tau}} f_1 \, p(t) - f_1(p, 0)$$

$$= \frac{1}{\left(i\omega + \frac{1}{\tau}\right)} \left(e \vec{E}_0 \cdot \vec{v} \frac{\partial f_0}{\partial \varepsilon_p}\right)\left(e^{\frac{t}{\tau} + i\omega t} - 1\right)$$

$$= \frac{e\tau}{1 + i\omega\tau} \left(e \vec{E}_0 \cdot \vec{v} \frac{\partial f_0}{\partial \varepsilon_p}\right)\left(e^{\frac{t}{\tau} + i\omega t} - 1\right)$$

$$f_1(\vec{p}, t)$$

$$= f_1(\vec{p}, 0) e^{-\frac{t}{\tau}}$$

$$+ \frac{1}{\left(i\omega + \frac{1}{\tau}\right)} \left(e \vec{E}_0 \cdot \vec{v} \frac{\partial f_0}{\partial \varepsilon_p}\right)\left(e^{i\omega t} - e^{-\frac{t}{\tau}}\right)$$

We now assume $f_1(\vec{p},0)$ to be some known function.

Note that the factor $e^{-t/\tau}$ vanish or are negligibly small after a few relaxation times $\tau \sim 10^{-10}$ s

c) In the following, we ignore $e^{-t/\tau}$ - terms, and consider $t \sim c\tau$ where $c \sim 2, 3$ or so.

$$f_1(\vec{p},t) \simeq \frac{1}{i\omega + \frac{1}{\tau}} \, e\vec{E}_0 \cdot \vec{v} \, \frac{\partial f_0}{\partial \varepsilon_p} \, e^{i\omega t}$$

$$= G(t)\tau e \, \vec{E}_0 \cdot \vec{v} \, \frac{\partial f_0}{\partial \varepsilon_p}$$

$$J_Q = 2\int \frac{d^3p}{(2\pi\hbar)^3} \, \vec{v}\,(\varepsilon_p - \mu) \, f_1(\vec{p},t)$$

$$J_e = -2e\int \frac{d^3p}{(2\pi\hbar)^3} \, \vec{v} \, f_1(\vec{p},t)$$

$$G(t) = \frac{1}{i\omega\tau+1} e^{i\omega t}$$

$\omega = 0$:

$$G(t) = 1$$

Thus expressions must back to the static case.

$$\vec{j_a} = 2G(t)\,\tau e \vec{E}_0 \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\vec{v}^2}{3} \frac{\partial f_0}{\partial \varepsilon_p} (\varepsilon_p - \mu)$$

$$\vec{j_e} = -2G(t)\,\tau e^2 \vec{E}_0 \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{v^2}{3} \frac{\partial f_0}{\partial \varepsilon_p}$$