Exam TFY4275/FY8907
28. 11. 2024

1a) $P_n(t)$: Conditional probability (transition probability) of being in state n at time t, given an starting state

$\omega_{nn'}$: Transition rate from n' to n.

b) $\omega_{nn'} = \lambda(n') \delta_{n',n+1} + \varkappa(n') \delta_{n',n-1}$

$$\frac{dP_n}{dt} = \lambda(n+1) P_{n+1} + \varkappa(n-1) P_{n-1} - [\lambda(n) + \varkappa(n)] P_n$$

c) $\lambda(n) = \rho \; ; \; \varkappa(n) = \eta$

$$\frac{dP_n}{dt} = \rho(P_{n+1} - P_n) + \eta(P_{n-1} - P_n)$$

$$P_n(0) = \delta_{n,0}$$

$$F(z,t) = \sum_n z^n P_n(t)$$

$$\frac{\partial F}{\partial t} = \sum_n z^n \frac{dP_n}{dt}$$

$$= \sum_n z^n \left[\rho(P_{n+1} - P_n) + \eta(P_{n-1} - P_n)\right]$$

$$= \sum_n \Big[\rho \left(z^{n-1} - z^n\right) P_n + \eta \left(z^{n+1} - z^n\right) P_n \Big]$$

$$= \sum_n \left[\rho\left(z^{-1} - 1\right) + \eta(z-1) \right] z^n P_n$$

$$\frac{\partial F}{\partial t} = \left[\rho\left(z^{-1} - 1\right) + \eta(z-1) \right] F(z,t)$$

$$z \frac{\partial F}{\partial t} = \left[\rho(1-z) + \eta z(z-1) \right] F$$

$$F(z,0) = \sum_n z^n P_n(0)$$

$$= \sum_n z^n \delta_{n,0} = 1$$

d)

$$\frac{dF}{dt} = A(z) F \; ; \; A(z) = \rho\left(z^{-1} - 1\right) + \eta(z-1)$$

$$\frac{dF}{F} = A(z)\, dt$$

$$F(z,t) = F(z,0)\, e^{A(z)t}$$

$$= e^{A(z)t}$$

$$= e^{-(\eta+\rho)t}\, e^{\frac{\rho t}{z}}\, e^{\eta z t} \; ; \; H(x) = e^x$$

e) $$F(z,t) = \sum_n \quad {}^{n} P_n(t)$$

$$\frac{\partial F}{\partial z} = \sum_n n\, z^{n-1} P_n(t)$$

$$\frac{\partial^2 F}{\partial z^2} = \sum_n n(n-1)\, z^{n-2} P_n(t)$$

$\underline{z=1:}$

$$\left.\frac{\partial F}{\partial z}\right|_{z=1} = \sum_n n P_n(t) = \langle n \rangle$$

$$\left.\frac{\partial^2 F}{\partial z^2}\right|_{z=1} = \sum_n n(n-1) P_n(t) = \langle n(n-1) \rangle$$

$$= \langle n^2 \rangle - \langle n \rangle$$

$$\langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2$$

c) $$\langle n \rangle = \left.\frac{\partial F}{\partial z}\right|_{z=1}$$

$$\frac{\partial F}{\partial z} = e^{-(\eta+\rho)t} \left(\eta t - \frac{\rho t}{z^2} \right) e^{\frac{\rho t}{z} + \eta z t}$$

$$\frac{\partial^2 F}{\partial z^2}$$

$$= e^{-(\eta+\rho)t}\left(\eta t - \frac{\rho t}{z^2}\right)^2 e^{\frac{\rho t}{z}+\eta z t}$$

$$+ e^{-(\eta+\rho)t}\,\frac{2\rho t}{z^3}\,e^{\frac{\rho t}{z}+\eta z t}$$

<u>$z=1$</u>:

$$\left.\frac{\partial F}{\partial z}\right|_{z=1} = \underline{\underline{(\eta-\rho)t}} = \langle n\rangle$$

$$\left.\frac{\partial^2 F}{\partial z^2}\right|_{z=1} = (\eta-\rho)^2 t^2 + 2\rho t$$

$$= \langle n^2\rangle - \langle n\rangle$$

$$\langle n^2\rangle - \langle n\rangle^2 = \langle n^2\rangle - \langle n\rangle + \langle n\rangle - \langle n\rangle^2$$

$$= (\eta-\rho)^2 t^2 + 2\rho t + (\eta-\rho)t - (\eta-\rho)^2 t^2$$

$$= \underline{\underline{(\eta+\rho)t}}$$

This is an asymmetric random walk, alternatively a diffusive drift-process

2a) RHS: Multiply by $A(\vec{v})$ and integrate over $\vec{v}$ to obtain RHS.

LHS: For each term, multiply by $A(\vec{v})$ and integrate over $\vec{v}$, then use definition of velocity-average, given in the problem.

i) $\int d^3v\, A \frac{\partial f}{\partial t}$

$$= \frac{\partial}{\partial t}\left(\int d^3v\, A f\right) = \frac{\partial}{\partial t}\left(n\langle A\rangle\right)$$

ii) $\int d^3v\, A\, \vec{v}\cdot\frac{\partial f}{\partial \vec{r}}$

$$= \frac{\partial}{\partial x_i}\int d^3v\, A\, v_i\, f$$

$$= \frac{\partial}{\partial x_i}\left(n\langle A v_i\rangle\right)$$

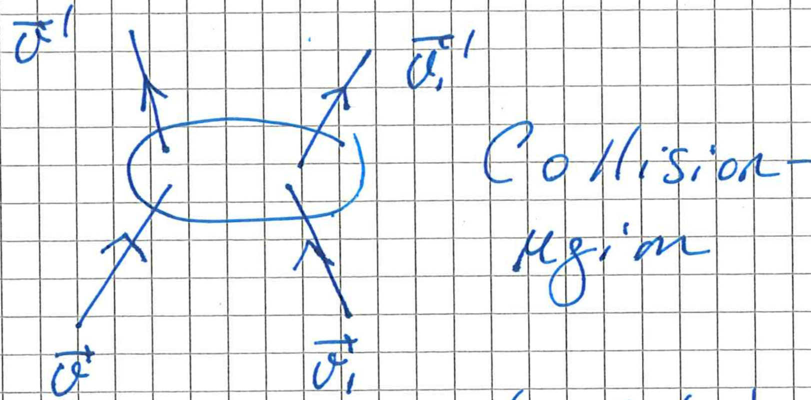
iii) $\int d^3v\, a_i \frac{\partial f}{\partial v_i} A$

$$= a_i \int d^3v\, A \frac{\partial f}{\partial v_i}$$

$$= -a_i \int d^3v\, \frac{\partial A}{\partial v_i} f$$

$$= -a_i\, n\left\langle\frac{\partial A}{\partial v_i}\right\rangle$$

b) Collision invariant $A(\vec{v})$



Collision-invariant $A(\vec{v})$ satisfies

$$A(\vec{v}) + A(\vec{v}_1)$$
$$= A(\vec{v}') + A(\vec{v}_1')$$

$$I = \int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\, g\, (f' f_1' - f f_1)\, A(\vec{v})$$

I is invariant under $\vec{v} \rightleftarrows \vec{v}_1$

$g = |\vec{v} - \vec{v}_1| = |\vec{v}_1 - \vec{v}|$

$\sigma(\Omega)$ is invariant under this relabelling

$f f_1 = f_1 f$

Thus:

(*) $$I = \frac{1}{2} \int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\, g\, (f' f_1' - f f_1) \cdot (A(\vec{v}) + A(\vec{v}_1))$$

Furthermore

$d^3v\, d^3v_1'$ due to elastic collisions. $\sigma(\Omega) = \sigma'(\Omega)$ (inverse process)

$g' = g$ (Elastic coll.) $= |\vec{v}' - \vec{v}_1'|$
$|\vec{v} - \vec{v}_1|$

$\cdot\ \vec{v} \rightleftarrows \vec{v}' \quad ; \quad \vec{v}_1' \leftrightarrow \vec{v}_1 .$

(xx) $$I = \frac{1}{2}\int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\, g \cdot \left(f f_1 - f' f_1'\right)\left(A(\vec{v}') + A(\vec{v}_1')\right)$$

Adding (x) + (xx) and dividing by 2!

$$I = \int d^3v \int d^3v_1 \int d\Omega\, \sigma(\Omega)\, g \left(f' f_1' - f f_1\right)\left[A(\vec{v}) + A(\vec{v}_1) - A(\vec{v}') - A(\vec{v}_1')\right]$$

If A is a collision-inv $\Rightarrow$

$I = 0$

since $A(\vec{v}) + A(\vec{v}_1) - A(\vec{v}') - A(\vec{v}_1') = 0$

---

c) $A = p_i$

Momentum-conservation means that $\vec{p}$ is a collision-invariant.

$$\frac{\partial}{\partial t}(n\, p_i) + \frac{\partial}{\partial x_j}\left(n \langle p_i \quad \rangle\right) - a_j\, n\, m\, \delta_{ij} = 0$$

$\rho = n m \qquad p_i = m v_i$

$$\frac{\partial}{\partial t}\left(\rho\langle v_i\rangle\right)+\frac{\partial}{\partial x_j}\left(\rho\langle v_i v_j\rangle\right)$$

$$-\underbrace{\rho\, a_i}_{=F_i} = 0$$

External force

$\langle v_i v_j\rangle$: $\qquad u_i = \langle v_i\rangle$

$$v_i = \delta v_i + u_i \;;\; \delta v_i = v_i - \langle u_i\rangle$$

$$\langle v_i v_j\rangle = \langle (u_i+\delta v_i)(u_j+\delta v_j)\rangle$$

$$= u_i u_j + \langle \delta v_i\, \delta v_j\rangle$$

$$+ u_i \underbrace{\langle \delta v_j\rangle}_{=0} + u_j \underbrace{\langle \delta v_i\rangle}_{=0}$$

$$\frac{\partial(\rho u_i)}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho\left(u_i u_j\right)\right)$$

$$= F_i - \frac{\partial}{\partial x_j}\left(\rho\langle \delta v_i\, \delta v_j\rangle\right) \quad \text{xxx}$$

d)

$$P_{ij} = \rho\langle \delta v_i\, \delta v_j\rangle$$

xxx is Newton's 2nd law for a fluid-element.

ρu_i: Momentum density

$\rho u_i u_j$: Momentumdensity current

F_i: External force

$$\frac{\partial}{\partial x_i}\left(\rho \left\langle \delta v_i \, \delta v_j \right\rangle\right):$$

Force acting on a fluid-element from its surrounding fluid. Contains in principle both hydrostatic pressure as well as frictional forces.

e) Given Ansatz

$$f = n \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{m}{2k_B T}(\vec{v}-\vec{u})^2}$$

NB! Note that this f is invariant under time-reversal, since $t \to -t \Rightarrow \vec{v} \to -\vec{v}$, $\vec{u} \to -\vec{u}$, and all other factors are invariant.

Define $\delta v_i = v_i - u_i = c_i$ to simplify notation

$$\Rightarrow d^3 v_i = d^3 c_i$$

$$P_{ij} = \rho \langle \delta u_i \, \delta u_j \rangle$$

$$= \rho n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int d^3u \; u_i u_j \, e^{-\alpha u^2}$$

$$\alpha = \frac{m}{2 k_B T}$$

Because of spherical symmetry, we only get a non-zero contribution if $i = j$.

$$P_{ij} = \rho n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{\delta_{ij}}{3} \cdot \int d^3u \; u^2 e^{-\alpha u^2}$$

We have:

$$\int d^3u \; u^2 e^{-\alpha u^2}$$

$$= 4\pi \int_0^\infty du \; u^4 e^{-\alpha u^2}$$

$$= 4\pi \, \frac{3}{8} \sqrt{\frac{\pi}{\alpha}} \, \frac{1}{\alpha^2}$$

$$P_{ij} = \rho n \left(\frac{\alpha}{\pi}\right)^{3/2} \frac{\delta_{ij}}{3} \frac{3}{8} 4\pi \sqrt{\frac{\pi}{\alpha}} \frac{1}{\alpha^2}$$

$$= \rho n \frac{\alpha}{\pi} \frac{1}{\alpha^2} \frac{\delta_{ij}}{2} \pi$$

$$= \rho n \delta_{ij} \frac{1}{2\alpha}$$

$$= m n \delta_{ij} \frac{k_B T}{m}$$

$$= \delta_{ij} n k_B T = \delta_{ij} p$$

$p = n k_B T$ (Ideal gas law)

Comparing with the general expression for P_{ij}, we see:

$$\eta = 0 \; ; \; \zeta = 0$$

f)
$$P_{ij} = \delta_{ij}\, p - \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\, \nabla\cdot\vec{u}\right) - \zeta\, \nabla\cdot\vec{u}$$

The hydrostatic pressure p is invariant under $t \to -t$

The two last terms are not invariant under time-reversal $t \to -t$ if $\eta \neq 0$, $\zeta \neq 0$.

In e), we computed

$$P_{ij} = \rho \langle u_i u_j \rangle$$

$$\left.\begin{array}{l} u_i \to -u_i \\ u_j \to -u_j \end{array}\right\} t \to -t$$

$$\Rightarrow \quad u_i u_j \to u_i u_j \,; \; t \to -t.$$

Recall that

$$\langle u_i u_j \rangle = \frac{1}{n} \int d^3u \, u_i u_j f$$

The only way of getting $\eta \neq 0$; $\zeta \neq 0$ such that P_{ij} is not TRI, is if f itself is not time-reversal invariant. (TRI)

But our Ansatz for f is TRI, and hence ~~we~~ P_{ij} with this f must be TRI, and hence $\eta = 0$; $\zeta = 0$.

∴ We could have noticed that $\eta = 0$; $\zeta = 0$ with this Ansatz for f purely on symmetry-grounds, without doing a calculation.