NTNU
Faculty of Natural Sciences

NTNU, Department of Physics

Exam TFY4345 Classical Mechanics Fall 2024

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU

Saturday November 30 2024 09:00–13:00

Problem 1

Fig. 1 shows a bead of mass m on rotating circular wire with constant angular speed ω. The radius of the wire is R. The gravitational acceleration g is directed downwards. We will be using spherical coordinates (r, ϕ, θ).

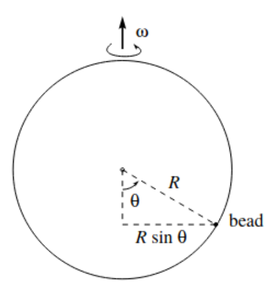

Figure 1: Rotating hoop with a bead of mass m.

a) There are two constraints in this problem

$$r = R\,, \tag{1}$$
$$\dot{\phi} = \omega\,. \tag{2}$$

Explain why the second constraint $\dot{\phi} = \omega$ is semi-holonomic. Explain why this constraint is equivalent to the following holonomic constraint

$$\phi = \omega t\,. \tag{3}$$

b) Using the method of Lagrange multipliers to enforce the constraints, show that the Lagrangian of the system can be written as

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right] + mgr\cos\theta\; - \lambda_1(r-R) - \lambda_2(\phi - \omega t)\,. \tag{4}$$

What is the interpretation of λ_1 and λ_2?

c) Calculate the Euler-Lagrange equations. Find explicit expressions for λ_1 and λ_2.

d) Calculate

$$\frac{dH}{dt}\,, \tag{5}$$

where H is the Hamiltonian of the system. Interpret the result.

e) Instead of using the method of Lagrange multipliers, we can enforce the constraint by using a single generalized coordinate, namely θ. Show that the Lagrangian can be written as

$$L = \frac{1}{2}mR^2\left[\sin^2\theta\omega^2 + \dot{\theta}^2\right] + mR^2\omega_0^2\cos\theta\,, \tag{6}$$

where we have introduced the natural angular speed $\omega_0 = \sqrt{\frac{g}{R}}$.

f Write down the equation of motion for θ.

g) Rewrite the equation of motion as a set of two coupled differential equations for θ and $p = \dot{\theta}$.

h) Determine the fixed points (θ^*, p^*) as functions of the angular speed ω. Calculate the eigenvalues of the Jacobian matrix J for each fixed point and classify the fixed point. **Hint:** Fig. 2 in the Appendix. Sketch the fixed points θ^* as functions of the dimensionless variable ω/ω_0. Any comments?

Problem 2

The Lagrangian for a particle with rest mass m moving in one dimension is

$$L = -mc\sqrt{-\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} + amB_{\mu\nu}\dot{x}^{\mu}x^{\nu} \ , \tag{7}$$

where $\mu = 0, 1$ and $\nu = 0, 1$, $B_{\mu\nu}$ is a covariant tensor of rank two, and a is a constant. The initial conditions are $x(\tau = 0) = u(\tau = 0) = 0$. We also set $t(\tau = 0) = 0$ for simplicity.

a) Is the Lagrangian L Lorentz invariant?

b) In the laboratory frame, the tensor $B_{\mu\nu}$ is the 2×2 matrix

$$B_{\mu\nu} = \frac{1}{c}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{8}$$

Write down the Lagrangian L in the laboratory frame.

c) Derive the equations of motion. Solve the equations of motion for $x(\tau)$ and $t(\tau)$. Any comments? **Hint:** use

$$\frac{\partial\left[-mc\sqrt{-\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}\right]}{\partial\dot{x}^{\alpha}} = m\dot{x}_{\alpha} \ . \tag{9}$$

d) Find the conserved quantity and interpret the result.

Problem 3

a) Formulate Kepler's three laws of planetary motion.

b) What does it mean that two planets are tidally locked to each other?

c) Define tidal force. It takes the Earth 24h50min to complete a rotation around its own axis. How often do high tide and low tide occur? Disregard the Sun and other planets in the Solar System.

Appendix

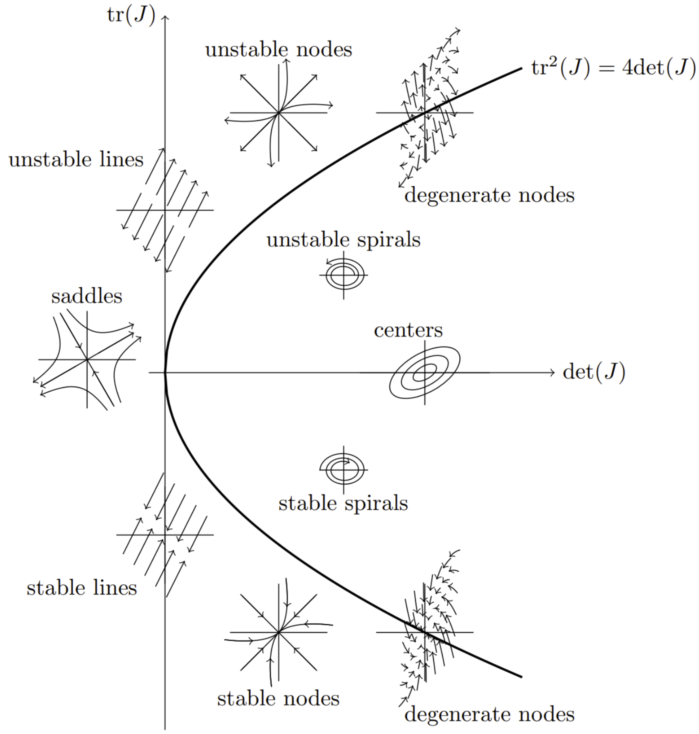

Figure 2: Classification of fixed points.