NTNU
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Solutions TFY4345 Classical mechanics Fall 2024

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Problem 1

a) The second constraint is of the form

$$f(\dot{\phi}) \quad = \quad \dot{\phi} = \omega \; , \tag{1}$$

which involves the time derivative of the generalized coordinate ϕ. It is therefore semiholonomic. Integrating this equation yields

$$\phi \quad = \quad \omega t \; , \tag{2}$$

where we for convenience have chosen $C = 0$, where C is an integration constant. The constraint is now of the form

$$f(\phi) \quad = \quad \phi = \omega t \; , \tag{3}$$

which is holonomic since it only involves the coordinate itself. It is also time dependent.

b) The kinetic energy of the particle in spherical coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2) \ . \tag{4}$$

The potential energy is given by

$$V = -mgz = -mgr\cos\theta \ . \tag{5}$$

The constraints are enforced by introducing the Lagrange multipliers λ_1 and λ_2 giving the terms $\lambda_1(r-R)$ and $\lambda_2(\phi-\omega t)$. Adding T, V, and the constraint term, yields

$$L = \underline{\underline{\frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right] + mgr\cos\theta - \lambda_1(r-R) - \lambda_2(\phi-\omega t)}} \ . \tag{6}$$

The Lagrange multipliers λ_1 and λ_2 are generalized forces that enforce the constraints. The dimension of λ_1 is N and so can be interpreted as the force that ensures that distance from the bead to the origin is R. The dimension of λ_2 is nm, which is the dimension of torque. It is the torque required for circular motion in the x–y plane with constant speed.

c) The Euler-Lagrange equations for λ_1 and λ_2 are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\lambda}_1} - \frac{\partial L}{\partial\lambda_1} = r - R = 0 \ , \tag{7}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\lambda}_2} - \frac{\partial L}{\partial\lambda_2} = \phi - \omega t = 0 \ , \tag{8}$$

or $r = R$ and $\phi = \omega t$, as required. Moreover

$$\frac{\partial L}{\partial r} = mr\left[\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2\right] + mg\cos\theta - \lambda_1 = mR\left[\sin^2\theta\omega^2 + \dot{\theta}^2\right] + mg\cos\theta - \lambda_1 \ , \tag{9}$$

$$\frac{\partial L}{\partial\phi} = -\lambda_2 \ , \tag{10}$$

$$\frac{\partial L}{\partial\theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - mgr\sin\theta \ , \tag{11}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{r}} = m\ddot{r} = 0 \ , \tag{12}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = 2mr^2\dot{\phi}\sin\theta\cos\theta\dot{\theta} + 2r\dot{r}\sin^2\theta\dot{\phi} + mr^2\sin^2\theta\ddot{\phi} \ , \tag{13}$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mR^2\ddot{\theta} \ . \tag{14}$$

This yields

$$m\ddot{r} = mr\left[\sin^2\theta\dot{\phi}^2 + \dot{\theta}^2\right] - mg\cos\theta - \lambda_1 , \tag{15}$$

$$2mr^2\dot{\phi}\sin\theta\cos\theta\dot{\theta} + 2r\dot{r}\sin^2\theta\dot{\phi} + mr^2\sin^2\theta\ddot{\phi} = -\lambda_2 \ , \tag{16}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = mr^2\sin\theta\cos\theta\dot{\phi}^2 - mgr\sin\theta \ . \tag{17}$$

Using $r = R$, $d\dot{r} = \ddot{r} = 0$, and $\dot{\phi} = \omega$, we obtain

$$\lambda_1 = mr\left[\sin^2\theta\omega^2 + \dot{\theta}^2\right] - mg\cos\theta \,, \tag{18}$$

$$\lambda_2 = -2mR^2\omega\sin\theta\cos\theta\dot{\theta} \,, \tag{19}$$

$$\ddot{\theta} = \sin\theta\cos\theta\omega^2 - \frac{g}{R}\sin\theta \,. \tag{20}$$

d)

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = -\lambda_2\omega \,. \tag{21}$$

The term $\lambda_2 d\phi = \lambda_2\omega dt$ is the work done by the constraint force when the bead moves a distance $R\sin\theta d\phi$. The work per unit time is then equal to $\frac{dH}{dt}$.

e) This follows directly from the original Lagrangian by setting $\lambda_1 = \lambda_2 = \dot{r} = 0$, $r = R$, $\dot{\theta} = \omega$, and $\omega_0 = \sqrt{\frac{g}{R}}$.

f) This follows directly from Eqs. (11) and (14)

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \omega_0^2\sin\theta \,. \tag{22}$$

g) The first equation is simply $\dot{\theta} = p$ which is yields $\ddot{\theta} = \dot{p}$ and therefore

$$\dot{\theta} = p = f(\theta, p) \,, \tag{23}$$

$$\dot{p} = \omega^2\sin\theta\cos\theta - \omega_0^2\sin\theta = g(\theta, p) \,. \tag{24}$$

h) The fixed points are found by solving the equations $f(\theta, p) = g(\theta, p) = 0$. The first equation yields $p = 0$, while the second yields $\theta^* = 0$ or $\theta^* = \pm\arccos\frac{\omega_0^2}{\omega^2}$. [1] The latter exists only if $\omega > \omega_0$, i.e. if the wire rotates sufficiently fast. Thus the fixed points are $(0, 0)$, $(\pm\arccos\frac{\omega_0^2}{\omega^2}, 0)$ The Jacobian matrix is

$$J = \begin{pmatrix} 0 & 1 \\ \omega^2(\cos^2\theta - \sin^2\theta) - \omega_0^2\cos\theta & 0 \end{pmatrix} \,. \tag{25}$$

We first consider $(0, 0)$. This yields

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \omega^2 - \omega_0^2 & 0 \end{pmatrix} \,, \tag{26}$$

[1] There is also an unstable fixed point at $(\pi, 0)$. Moreover, I have not discussed the borderline cases where $\omega = \omega_0$.

whose eigenvalues are $\lambda_{a,b} = \pm\sqrt{\omega^2 - \omega_0^2}$. For $\omega < \omega_0$, the eigenvalues are purely imaginary and the fixed point is a **center**. For $\omega > \omega_0$, the eigenvalues are purely real and with opposite sign. Thus the fixed point is a **saddle**. We next consider the fixed points, whose Jacobian is

$$J(\pm \arccos \frac{\omega_0^2}{\omega^2}, 0) = \begin{pmatrix} 0 & 1 \\ 2\omega^2\left(\frac{\omega_0^4}{\omega^4} - 1\right) & 0 \end{pmatrix}, \tag{27}$$

with eigenvalues $\lambda_{a,b} = \sqrt{2}\omega\sqrt{\frac{\omega_0^4}{\omega^4} - 1}$. Since $\omega_0 < \omega$, the eigenvalues are purely imaginary and are **centers**.

The fixed point $(0, 0)$ exists for all values of $\omega > 0$, but changes from a center to a saddle at $\omega = \omega_0$, exactly at the point where two new fixed points emerge. This is shown in Fig. 1 and is called a **pitchfork bifurcation**.

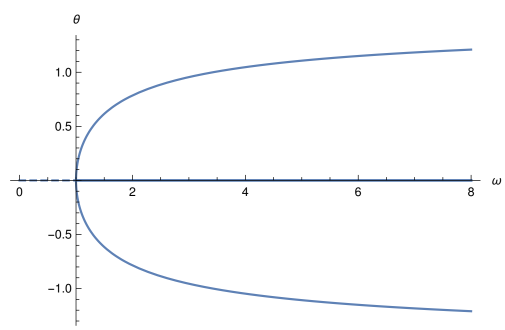

Figure 1: Fixed points θ^* as functions of ω/ω_0.

Problem 2

a) The first term is the square root of the contraction of a covariant and a contravariant vector, i.e a Lorentz scalar. The second term is the contraction of a contravariant tensor of rank two $B_{\mu\nu}$ and a covariant vector $\dot{x}^\mu x^\mu$ and is therefore a Lorentz scalar. The Lagrangian is therefore invariant under Lorentz transformations and a Lorentz scalar.

b) The only nonzero term is B_{01} giving $\frac{am}{c}\dot{x}^0 x^1$ for the second term and therefore

$$L = \underline{\underline{-mc\sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} + am\dot{t}x}}\ . \tag{28}$$

c) The partial derivatives are

$$\frac{\partial L}{\partial \dot{t}} = -mc^2\dot{t} + amx \ , \quad (29)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \ , \quad (30)$$

$$\frac{\partial L}{\partial t} = 0 \ , \quad (31)$$

$$\frac{\partial L}{\partial x} = am\dot{t} \ . \quad (32)$$

This yields the equations of motion

$$\frac{d}{d\tau}(mc^2\dot{t} - amx) = 0 \ , \quad (33)$$

$$\ddot{x} = a\dot{t} \ . \quad (34)$$

Upon integration, the first equation yields $c^2\dot{t} - ax = C$, where C is an integration constant. Since $u(0) = 0$, we have $\dot{t}(0) = 1$. Since $x(0) = 0$, we find $C = c^2$. Substituting this result into the second equation gives

$$\ddot{x} - \frac{a^2}{c^2}x = a \ . \quad (35)$$

This is a second-order linear inhomogenous differential equation whose solution is the general solution to the homogeneous equation and the particular solution to the inhomogeneous,

$$x(\tau) = A\cosh\frac{a}{c}\tau + B\sinh\frac{a}{c}\tau - \frac{c^2}{a} \ . \quad (36)$$

The initial conditions give $A - \frac{c^2}{a} = 0$ and $B = 0$, and so

$$x(\tau) = \underline{\underline{\frac{c^2}{a}\left[\cosh\frac{a}{c}\tau - 1\right]}} \ . \quad (37)$$

The first equation now reads

$$\dot{t} = \frac{a}{c^2}x + 1 \ . \quad (38)$$

Integration gives

$$t(\tau) = \frac{c}{a}\underline{\underline{\sinh\frac{a}{c}\tau}} \ , \quad (39)$$

where have used the initial condition $t(\tau) = 0$ to determine the integration constant. We recognize $x(\tau)$ and $t(\tau)$ as solutions to the problem of constant acceleration $g = a$ in the instantaneous rest frame or hyperbolic motion.

d) We have already shown that $mc^2\dot{t} - amx$ is a constant of motion. This is the consequence of ct being a cyclic coordinate. This constant was also shown to take the value mc^2, which is the rest mass energy.

Problem 3

a) Kepler's laws of planetary motion are

(a) Planetary motion is elliptical with the Sun in one of the foci

(b) A line joining the Sun and a planet sweeps out equal areas in equal times.

$$\frac{dA}{dt} = \underline{\underline{\frac{1}{2}\frac{L}{m}}} . \tag{40}$$

(c) The square of the period T is proportional to the cube of the semi-major axis a

$$\underline{\underline{T^2 \propto a^3}} . \tag{41}$$

b) It means that it takes each planet as long to rotate around its own axis as it does to revolve around its partner.

c) Tidal force refers to the variation in the gravitational field over a body generated by another body. The Moon's gravitational pull generates the tidal force. The tidal force causes Earth—and its water—to bulge out on the side closest to the Moon and the side farthest from the Moon. Thus high tide will be every 12h25min.