15 Gravity as a gauge theory

Aim of this short chapter is develop the formalism necessary to describe the interaction of fermions with gravity. Moreover, we want to stress the similarity of gravity as gauge theory with the group GL(4) to "usual" Yang-Mills theories. Finally, we want to understand how gravity selects among the many mathematically possible connections on a Riemannian manifold.

15.1 Vielbein formalism and the spin connection

For fields transforming as tensor under Lorentz transformation, the effects of gravity are accounted for by the replacements $\{\partial_\mu, \eta_{\mu\nu}\} \to \{\nabla_\mu, g_{\mu\nu}\}$ in the matter Lagrangian $\mathscr{L}_{\rm m}$ and the resulting physical laws. Imposing the two requirements $\nabla_\rho g_{\mu\nu} = 0$ ("metric connection") and ${\Gamma^\alpha}_{\beta\gamma} = {\Gamma^\alpha}_{\gamma\beta}$ ("torsionless connection") selects uniquely the Levi-Civita connection. In the following, we want to understand if these conditions are a consequence of Einstein gravity or necessary additional constraints.

The substitution rule $\{\partial_\mu, \eta_{\mu\nu}\} \to \{\nabla_\mu, g_{\mu\nu}\}$ cannot be applied to the case of spinor representations of the Lorentz group. Instead, we apply the equivalence principle as physical guide line to obtain the physical laws including gravity: More precisely, we use that we can find at any point P a local inertial frame for which the physical laws become those known from Minkowski space.

We demonstrate this first for the case of a scalar field ϕ. The usual Lagrange density without gravity,

$$\mathscr{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)\,, \tag{15.1}$$

is still valid on a general manifold $\mathcal{M}(\{x^\mu\})$, if we use in each point P locally free-falling coordinates, $\xi^a(P)$. In order to distinguish these two sets of coordinates, we label inertial coordinates by latin letters while we keep greek indices for arbitrary coordinates. We choose the locally free-falling coordinates ξ^a to be ortho-normal. Thus in these coordinates the metric is given by $\mathrm{d}s^2 = \eta_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b$ with $\eta = \mathrm{diag}(1,-1,-1,-1)$. Then the action of a scalar field including gravity is

$$S[\phi] = \int \mathrm{d}^4\xi \left[\frac{1}{2}\eta^{ab}\partial_a\phi\partial_b\phi - V(\phi)\right]\,, \tag{15.2}$$

where $\partial_a = \partial/\partial\xi_a$. This action looks formally exactly as the one without gravity – however we have to integrate over the manifold $\mathcal{M}(\{x^\mu\})$ and all effects of gravity are hidden in the dependence $\xi^a(x^\mu)$.

We introduce now the vierbein (or tetrad) fields e^a_α by

$$\mathrm{d}\xi^m = \frac{\partial\xi^m}{\partial x^\mu}\,\mathrm{d}x^\alpha \equiv \mathrm{e}^m_\mu(x)\,\mathrm{d}x^\mu\,. \tag{15.3}$$

Thus we can view the vierbein $e_\mu^m(x)$ both as the transformation matrix between arbitrary coordinates x and inertial coordinates ξ or as a set of four vectors in T_x^*M. In the absence of gravity, we can find in the whole manifold coordinates such that $e_\mu^m(x) = \delta_\mu^m$.

We define analogously the inverse vierbein e_m^μ by

$$\mathrm{d}x^\mu = \frac{\partial x^\mu}{\partial \xi^m}\,\mathrm{d}\xi^m \equiv e_m^\mu(x)\,\mathrm{d}\xi^m\,. \tag{15.4}$$

The name is justified by

$$\mathrm{d}\xi^m = e_\mu^m\,\mathrm{d}x^m = e_\mu^m e_n^\mu\,\mathrm{d}\xi^n\,, \tag{15.5}$$

i.e. $e_\mu^m e_n^\mu = \delta_n^m$. Moreover, we can view the vierbein as a kind of square-root of the metric tensor, since

$$\mathrm{d}s^2 = \eta_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b = \eta_{mn}e_\mu^m e_\nu^n \mathrm{d}x^\mu\mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{15.6}$$

and thus

$$g_{\mu\nu} = \eta_{mn}e_\mu^m e_\nu^n\,. \tag{15.7}$$

Taking the determinant of this equation, we see that the volume element is

$$\mathrm{d}^4\xi = \sqrt{|g|}\mathrm{d}^4x = \det(e_\mu^m)\mathrm{d}^4x \equiv E\mathrm{d}^4x\,. \tag{15.8}$$

For later use, we note that latin indices are raised and lowered by the flat metric. It is also possible to construct mixed tensors, having both latin and greek indices. For instance, we can rewrite the energy-momentum tensor as $T_{\mu\nu} = e_\mu^m T_{m\nu} = e_\mu^m e_\nu^n T_{mn}$.

We have now all the ingredients needed to express (15.2) in arbitrary coordinates x^μ of the manifold $\mathcal{M}$. We first change the derivatives,

$$\mathscr{L} = \frac{1}{2}\eta_{mn}e_\mu^m e_\nu^n \partial^\mu\phi\partial^\nu\phi - V(\phi) = \frac{1}{2}g_{\mu\nu}\partial^\mu\phi\partial^\nu\phi - V(\phi) \tag{15.9}$$

and then the volume element in the action,

$$S[\phi] = \int \mathrm{d}^4x\sqrt{|g|}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)\right]\,. \tag{15.10}$$

As it should, we reproduced the usual action of a scalar field including gravity. Note that the sole effect of gravitational interactions is contained in the metric tensor and its determinant, while the connection plays no role since $\nabla_\mu\phi = \partial_\mu\phi$. Similarly, the connection drops out of the Lagrangian of a Yang-Mills field, since the field-strength tensor is antisymmetric.

Fermions We now proceed to the spin-1/2 case. Without gravity, we have

$$\mathscr{L} = \bar\psi(\mathrm{i}\gamma^\mu\partial_\mu - m)\psi \tag{15.11}$$

with $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Performing a Lorentz transformation, $\tilde{x}^a = \Lambda^b{}_a x^b$, the Dirac spinor ψ transforms as

$$\tilde\psi(\tilde{x}) = S(\Lambda)\psi(x) = \exp\left(-\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\psi(x) \tag{15.12}$$

with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ as the six parameters and $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$ as the six infinitesimal generators of these transformations.

Switching on gravity, we replace $x^\mu \to \xi^m$ and $\gamma^\mu \to \gamma^m$. General covariance reduces now to the requirement that we have to allow in an inertial system arbitrary Lorentz transformations. The condition that the Dirac equation is invariant under local Lorentz transformations $\Lambda(x)$ allows us to derive the correct covariant derivative, in a manner completely analogous to the Yang-Mills case: We have to compensate the term introduced by the space-time dependence of $S(\xi)$ in

$$\partial_a \psi \to \tilde{\partial}_a \tilde{\psi}(\tilde{\xi}) = \Lambda^b{}_a \partial_b [S(\xi)\psi(\xi)] \tag{15.13}$$

by introducing a "latin" covariant derivative

$$\nabla_a = \mathrm{e}_a^\alpha \left(\partial_\alpha + \mathrm{i}\omega_\alpha\right) , \tag{15.14}$$

and requiring an inhomogeneous transformation law

$$\omega_\alpha \to \tilde{\omega}_\alpha = S\omega_\alpha S^\dagger - \mathrm{i}S\partial_\alpha S^\dagger \tag{15.15}$$

for ω. As a result,

$$\nabla_a \to \tilde{\nabla}_a = \Lambda^b{}_a S \nabla_a S^\dagger \tag{15.16}$$

and the Dirac Lagrangian is invariant under local Lorentz transformations.

The connection ω_α is a matrix in spinor space. Expanding it in the basis $\sigma^{\mu\nu}$, we find as a more explicit expression for the covariant derivative

$$\begin{aligned}\nabla_a &= \partial_a + \frac{\mathrm{i}}{2}\omega_a^{\mu\nu}\sigma_{\mu\nu} = \mathrm{e}_a^\alpha \left(\partial_\alpha + \mathrm{i}\frac{1}{2}\omega_\alpha^{\mu\nu}\sigma_{\mu\nu}\right) \\ &= \mathrm{e}_a^\alpha \left(\partial_\alpha + \frac{1}{2}\omega_\alpha^{\mu\nu}X_{\mu\nu}\right) .\end{aligned} \tag{15.17}$$

In the second step we replaced the infinitesimal generators $\sigma^{\mu\nu}$ specific for the spinor representation by the general generators $X^{\mu\nu}$ of Lorentz transformations chosen appropriate for the representation the ∇_a act on. The Lie algebra of the Lorentz group implies that the connection $\omega_a^{\mu\nu}$ is antisymmetric in the indices $\mu\nu$, if they are both up or down, $\omega_a^{\mu\nu} = -\omega_a^{\nu\mu}$.

The transformation law (15.15) of the spin connection ω under Lorentz transformations S is completely analogous to the transformation properties (8.13) of a Yang-Mills field A^μ under gauge transformations U. One should keep in mind however two important differences: First, a vector lives in a tangent space which is naturally associated to a manifold: In particular, we can associate a vector in $T_P M$ with a trajectory $x(\sigma)$ through P. Therefore we have the natural coordinate basis ∂_i in $T_P M$ and can introduce vielbein fields. In contrast, matter fields $\psi(x)$ lives in their group manifold which is attached arbitrarily at each point of the manifold, and the gauge fields act as a connection telling us how we should transport $\psi(x)$ to $\tilde{\psi}(x')$. However, there is no coordinate basis in the group manifold and nothing like torsion for the gauge fields. Second, we associate physical particles with spin s to the corresponding representations of the Poincaré group: Thus we identify fluctuations of the gauge field A^μ as the quanta of the vector field, while it is in the case of gravity not the fluctuation of the connection but of the metric tensor $g^{\mu\nu}$.

Transition to the standard notation We now establish the connection between the vierbein and the standard formalism, using that for tensor fields the two formalisms have to agree. Inserting into the definition of the covariant derivative for a vector A,

$$\nabla_\mu A^a = \partial_\mu A^a + \omega_\mu{}^a{}_b A^b , \tag{15.18}$$

the decomposition $A^a = \mathrm{e}^a_\nu A^\nu$ and requiring the validity of the Leibniz rule gives

$$\begin{aligned}\nabla_\mu A^a &= (\nabla_\mu \mathrm{e}^a_\nu) A^\nu + \mathrm{e}^a_\nu (\nabla_\mu A^\nu) \\ &= (\nabla_\mu \mathrm{e}^a_\nu) A^\nu + \mathrm{e}^a_\nu \left(\partial_\mu A^\nu + \Gamma^\nu{}_{\mu\lambda} A^\lambda \right) .\end{aligned} \tag{15.19}$$

Using

$$\partial_\mu A^a = \partial_\mu (\mathrm{e}^a_\nu A^\nu) = \mathrm{e}^a_\nu (\partial_\mu A^\nu) + (\partial_\mu \mathrm{e}^a_\nu) A^\nu . \tag{15.20}$$

to eliminate the second term in (15.19), we obtain

$$\nabla_\mu A^a = (\nabla_\mu \mathrm{e}^a_\nu) A^\nu + \partial_\mu A^a - (\partial_\mu \mathrm{e}^a_\nu) A^\nu + \mathrm{e}^a_\nu \Gamma^\nu{}_{\mu\lambda} A^\lambda . \tag{15.21}$$

Comparing to (15.18) we can read off how the covariant derivative acts on an object with mixed indices,

$$\nabla_\mu \mathrm{e}^a_\nu = \partial_\mu \mathrm{e}^a_\nu - \Gamma^\lambda{}_{\nu\mu} \mathrm{e}^a_\lambda + \omega_\mu{}^a{}_b \mathrm{e}^b_\nu \tag{15.22}$$

More generally, covariant indices are contracted with the usual connection $\Gamma^\lambda{}_{\nu\mu}$ while vierbein indices are contracted with $\omega_\mu{}^a{}_b$.

In a moment, we will show that the covariant derivative of the vielbein field is zero, $\nabla_\mu \mathrm{e}^a_\nu = 0$. Sometimes this property is called tetrade "postulate," but in fact it follows naturally from the definition of the vierbein field.

In order to derive an explicit formula for the spin connection $\omega_\mu{}^a{}_b$ we compare now the the covariant derivative of a vector in the two formalisms. First, we write in a coordinate basis

$$\boldsymbol{\nabla A} = (\nabla_\mu A^\nu) \mathrm{d}x^\mu \otimes \partial_\nu = (\partial_\mu A^\nu + \Gamma^\nu{}_{\mu\lambda} A^\lambda) \mathrm{d}x^\mu \otimes \partial_\nu . \tag{15.23}$$

Next we compare this expression to the one using a mixed basis,

$$\boldsymbol{\nabla A} = (\nabla_\mu A^m)\, \mathrm{d}x^\mu \otimes e_m = (\partial_\mu A^m + \omega_\mu{}^m{}_n A^n)\, \mathrm{d}x^\mu \otimes e_m \tag{15.24}$$

$$= [\partial_\mu (e^m_\nu A^\nu) + \omega_\mu{}^m{}_n e^n_\lambda A^\lambda]\, \mathrm{d}x^\mu \otimes (e^\sigma_m \partial_\sigma) . \tag{15.25}$$

Moving e^σ_m to the left and using the Leibniz rule as well as $e^\sigma_m e^m_\nu = \delta^\sigma_\nu$, it follows

$$\boldsymbol{\nabla A} = e^\sigma_m [e^m_\nu \partial_\mu A^\nu + A^\nu \partial_\mu e^m_\nu + \omega_\mu{}^m{}_n (e^n_\lambda A^\lambda)]\, \mathrm{d}x^\mu \otimes \partial_\sigma \tag{15.26}$$

$$= [\partial_\mu A^\sigma + e^\sigma_m (\partial_\mu e^m_\nu) A^\nu + e^\sigma_m e^n_\lambda \omega_\mu{}^m{}_n A^\lambda]\, \mathrm{d}x^\mu \otimes \partial_\sigma . \tag{15.27}$$

Thus

$$\Gamma^\nu{}_{\mu\lambda} = e^\sigma_m \partial_\mu e^m_\nu + e^\sigma_m e^n_\lambda \omega_\mu{}^m{}_n = e^m_\mu \nabla_\lambda e^\nu_m \tag{15.28}$$

or multiplying with two vierbein fields,

$$\omega_\mu{}^m{}_n = e^\sigma_m e^n_\lambda \Gamma^\nu{}_{\mu\lambda} - e^\sigma_m \partial_\mu e^m_\nu . \tag{15.29}$$

Clearly, the connection $\Gamma^\nu{}_{\mu\lambda}$ is in general not symmetric, $\Gamma^\nu{}_{\mu\lambda} \neq \Gamma^\nu{}_{\lambda\mu}$.

Next we show that the covariant derivative of the vierbein vanishes. The covariant derivative of a general mixed two-tensor is given by

$$\nabla_\rho X_{\nu p} = \partial_\rho X_{\nu p} + \Gamma^\lambda{}_{\nu\mu} X_{\nu p} + \omega_\mu{}^a{}_b X_{\nu p} . \tag{15.30}$$

Setting now $X_{\nu p} = \eta_{pq} e^q_\nu$ and expressing the connection via Eq. (15.28), it follows

$$\nabla_\rho X_{\nu p} = \nabla_\rho e_{\nu p} = \partial_\rho \mathrm{e}_{\nu p} + \omega_{\rho p \nu} + e_{\sigma p} e^m_\nu \nabla_\rho e^\sigma_m = \tag{15.31}$$

$$= \partial_\rho \mathrm{e}_{\nu p} - \partial_\rho \mathrm{e}_{\nu p} + \omega_{\rho p \nu} + \omega_{\rho \nu p} = 0 , \tag{15.32}$$

because of the antisymmetry of the spin connection. Similarly, one shows that $\nabla_\rho e^{\nu p} = 0$.

15.2 Action of gravity

Analogy to the YM case suggests as Lagrange density for the gravitational field

$$\mathscr{L} = \sqrt{|g|} R_{abcd} R^{abcd} \,. \tag{15.33}$$

This Lagrange density has dimension 4 and would thus lead to dimensionless gravitational coupling constant, in contradiction to Newton's law. Hilbert chose instead the curvature scalar which has the required mass dimension $d = 6$,

$$\mathscr{L}_{\rm EH} = \sqrt{|g|} R \,. \tag{15.34}$$

As we know, we can always add a constant term to the Lagrangian, $R \to R - 2\Lambda$, which would act as cosmological constant.

If we would consider gravity coupled to fermions, we would have to use the spin connection ω_μ in the matter Lagrangian as well as expressing $\sqrt{|g|}$ and R through latin quantities, $\sqrt{|g|} \to E$ and $R = R_{\mu\nu} g^{\mu\nu} \to R_{mn} \eta^{mn}$.

Einstein-Hilbert action The action (15.34) was introduced by Hilbert as a functional of the metric tensor g_{ab}: The Lagrangian is a function of the metric, its first and second derivatives, $\mathscr{L}_{\rm EH}(g_{ab}, \partial_c g_{ab}, \partial_c \partial_d g_{ab})$, and a variation of the action with respect to the metric gives the field equations.

Palatini action We start from the Einstein-Hilbert action (15.34), but consider it now as a functional of the metric tensor g_{ab}, the connection and its first derivatives, $\mathscr{L}_{\rm EH}(g_{ab}, \Gamma^a{}_{bc}, \partial_d \Gamma^a{}_{bc})$, where the metric tensor and the connection are varied independently,

$$\mathscr{L}_{\rm EH} = \sqrt{|g|} g^{\mu\nu} R_{\mu\nu} = \sqrt{|g|} g^{\mu\nu} \left(\partial_\nu \Gamma^\sigma{}_{\mu\sigma} - \partial_\sigma \Gamma^\sigma{}_{\mu\nu} + \Gamma^\tau{}_{\mu\sigma} \Gamma^\sigma{}_{\tau\sigma} - \Gamma^\tau{}_{\mu\nu} \Gamma^\sigma{}_{\tau\sigma} \right) \,. \tag{15.35}$$

The variation with respect to the metric,

$$\delta_g S_{\rm EH} = \int_\Omega {\rm d}^4 x \; \delta \left\{ \sqrt{|g|} \, g^{\mu\nu} \right\} R_{\mu\nu} = 0 \tag{15.36}$$

gives $R_{\mu\nu} = 0$, i.e. the usual Einstein equations in vacuum.

For the variation with respect to the connection we use as usually first the Palatini equation,

$$\delta_\Gamma S_{\rm EH} = \int_\Omega {\rm d}^4 x \sqrt{|g|} \, g^{\mu\nu} \delta R_{\mu\nu} = \int_\Omega {\rm d}^4 x \sqrt{|g|} \, g^{\mu\nu} \left[\nabla_\nu (\delta \Gamma^\sigma{}_{\mu\sigma}) - \nabla_\sigma (\delta \Gamma^\sigma{}_{\mu\nu}) \right] \tag{15.37}$$

Using then Leibiz rule and relabelling some indices, we find

$$\begin{aligned} \delta_\Gamma S_{\rm EH} = & \int_\Omega {\rm d}^4 x \sqrt{|g|} \nabla_\nu \left[g^{\mu\nu} \delta \Gamma^\sigma{}_{\mu\sigma} - g^{\mu\sigma} \delta \Gamma^\sigma{}_{\mu\sigma} \right] \\ & + \int_\Omega {\rm d}^4 x \sqrt{|g|} \left[(\nabla_\nu g^{\mu\nu}) \delta \Gamma^\sigma{}_{\mu\sigma} - (\nabla_\nu g^{\mu\sigma}) \delta \Gamma^\sigma{}_{\mu\sigma} \right] \end{aligned} \tag{15.38}$$

We kept the second line, because we consider an arbitrary connection. Thus we do not know yet if the covariant derivative of the metric vanishes.

Next we perform a partial integration of the first two terms, converting it into a surface term which we can drop. In the remaining part we relabel indices so that we can factor out the variation of the connection,

$$\delta_\Gamma S_{\mathrm{EH}} = -\int_\Omega \mathrm{d}^4x\sqrt{|g|}\left[\delta^\nu_\rho \nabla_\sigma g^{\mu\sigma} - \nabla_\rho g^{\mu\nu}\right]\delta\Gamma^\rho{}_{\mu\nu} \tag{15.39}$$

We assume now that the connection is symmetric. Then also the variation $\delta\Gamma^\rho{}_{\mu\nu}$ is symmetric and the antisymmetric part in the square bracket drops out. Asking that $\delta_\Gamma S_{\mathrm{EH}} = 0$ gives therefore

$$\frac{1}{2}\delta^\nu_\rho \nabla_\sigma g^{\mu\sigma} + \frac{1}{2}\delta^\mu_\rho \nabla_\sigma g^{\nu\sigma} - \nabla_\rho g^{\mu\nu} = 0 \tag{15.40}$$

or $\nabla_\sigma g^{\mu\nu} = \nabla_\sigma g_{\mu\nu} = 0$. Thus the Einstein-Hilbert action implies the metric compatibility of the connection.

Performing the same exercise with the Einstein-Hilbert plus the matter action considered as functional of the vierbein e^μ_m and the connection ω_μ, one finds the following: From the variation $\delta_\omega S_{\mathrm{EH}}$ one obtains automatically a metric connection which is however in general not symmetric. The torsion is sourced by the spin-density of fermions. Since in practically all macroscopic systems the spin-density of fermions is negligible, the use of a symmetric connection is thus justified. The variation $\delta_e S_{\mathrm{EH}}$ gives the usual Einstein equation.

15.3 Wess-Zumino model and supersymmetry

One of the primary motivations of supersymmetry is that we have symmetries which applies between fermions and other fermions, we have symmetries between bosons and other bosons, but none connected. We have no way of unifying them like like we do leptons with quarks at the grand unification scale (if that is the case). We will therefore here look at a theory where such a symmetry exists.

Taking the simplest case we will look at a single massless fermion and some scalar fields. An on-shell massless fermion has two degrees of freedom, and we will therefore need two scalar fields to match this. We still want the theory to hold off-shell so that we can handle virtual particles in Feynman diagrams, but we will deal with that later. The simple Lagrangian then reads:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu S \partial^\mu S + \frac{1}{2}\partial_\mu P \partial^\mu P + \frac{1}{4}\bar{\chi}\gamma^\mu \overleftrightarrow{\partial}_\mu \chi \tag{15.41}$$

Where we have used the $\overleftrightarrow{\partial}$ to make it a little bit more symmetric. A partial integration will yield the standard Lagrangian.

$$\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$$

Some technical remarks to keep in mind for the following calculations:

i. We use Majorana spinors. That is the χ are Grassmannian variables.

ii. Majorana spinors is written $\chi = \left(\begin{smallmatrix}\xi \\ -i\sigma^2\xi\end{smallmatrix}\right)$, where χ is a 4-component spinor, and ξ is a 2-component one. So the Majorana spinors only have two degrees of freedom like the Weyl-spinors. Two Majorana spinors (ξ and η) can be shown to satisfy