1.12 Wave equation for a string.
Consider a string of length L, mass density ρ and tension κ in one spatial dimension. Denoting its deviation from its equilibrium position x_0 with $\phi(x,t) \equiv x(t) - x_0$, write down its kinetic and potential energy (density) and the corresponding action (for small oscillations). Derive its equation of motion. [Note: $\phi(x,t)$ depends on t and x, and the Lagrange equation for $L(\phi, \partial_t\phi, \partial_x\phi)$ will contain $\mathrm{d}/\mathrm{d}t$ and $\mathrm{d}/\mathrm{d}x$ terms.]

The kinetic and potential energy density of the string are

$$T = \int_0^L \mathrm{d}x \, \frac{\rho}{2}(\partial_t\phi)^2 \quad \text{and} \quad V = \int_0^L \mathrm{d}x \, \frac{\kappa}{2}(\partial_x\phi)^2 \, .$$

[In general, V contains also higher derivatives which can become important for large oscillations.] The corresponding Lagrange function $L = T - V$, Lagrange density $\mathscr{L}$, and action are

$$S = \int \mathrm{d}t L = \int \mathrm{d}t \int_0^L \mathrm{d}x \, \mathscr{L} = \int \mathrm{d}t \int_0^L \mathrm{d}x \, \mathscr{L} = \int \mathrm{d}t \int_0^L \mathrm{d}x \, \left[\frac{\rho}{2}(\partial_t\phi)^2 - \frac{\kappa}{2}(\partial_x\phi)^2\right]$$

The Lagrange equations are

$$0 = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_t \frac{\partial\mathscr{L}}{\partial(\partial_t\phi)} - \partial_x \frac{\partial\mathscr{L}}{\partial(\partial_x\phi)}$$

or

$$0 = 0 - \rho\frac{\partial^2\phi}{\partial t^2} + \kappa\frac{\partial^2\phi}{\partial x^2} \, .$$

This a wave equation, $(1/v^2)\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$ with wave speed $c = \sqrt{\kappa/\rho}$.