1.3 Higher derivatives.

a.) Varying the action for $L = L(q, \dot{q}, \ddot{q}, \dots)$ gives

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q} + \cdots \right] = \tag{1}$$

$$= \int_{t_1}^{t_2} \mathrm{d}t \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} - \dots \right] \delta q. \tag{2}$$

In general, a term with a n.th time derivative of q has to be n times partially integrated, giving the term

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} \frac{\partial L}{\partial(\partial^n q/\partial t^n)}$$

in the Euler-Lagrange equation.

b.) The Euler-Lagrange equation for $L = L(q, \dot{q}, \ddot{q})$ is

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} = 0. \tag{3}$$

If L is not a degenerated function of $\ddot{q}$, i.e. if $\partial^2 L/\partial \ddot{q}^2 \neq 0$, then the Euler-Lagrange equation is of fourth order in q and thus four initial conditions are required: $q(t_1)$, $\dot{q}(t_1)$, $\ddot{q}(t_1)$ and $\dddot{q}(t_1)$. Therefore, four canonical coordinates are needed.
We assume that L has no explicit time-dependence. Then time-translation invariance implies conservation of energy and $\partial L/\partial t = 0$. Using then the Euler-Lagrange equation (3), it follows

$$0 = \frac{\partial L}{\partial t} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q} \tag{4}$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{q}} \right) \dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} + \frac{\partial L}{\partial \ddot{q}}\dddot{q} \tag{5}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} \right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}} \right) \dot{q} \right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}}\ddot{q} \right). \tag{6}$$

Thus

$$H = \left(\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}} \right) \dot{q} + \frac{\partial L}{\partial \ddot{q}}\ddot{q} - L(q, \dot{q}, \ddot{q}) \tag{7}$$

is conserved and can be identified with the Hamiltonian. This suggests the following choice for the canonical variables,

$$Q_1 = q, \qquad P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}} \right), \tag{8}$$

$$Q_2 = \dot{q}, \qquad P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{9}$$

Now we have to show that

$$H(P_1, P_2, Q_1, Q_2) = P_1 \dot{Q}_1 + P_2 \dot{Q}_2 - L$$

where $\ddot{q}$ should be expressed as function of the P_i and the Q_i leads to the correct time evolution. The assumption of non-degeneracy gurantees that we can solve for

$$\ddot{q} = f(Q_1, Q_2, P_2).$$

Here, f can be choosen to be independent of e.g. P_1, because $L = L(q, \dot{q}, \ddot{q})$ depends only ony three variables.
The first Hamilton equation is trivial,

$$\frac{\partial H}{\partial P_1} = \dot{Q}_1.$$

In the next one we use the chain rule and identify $\dot{Q}_2$ with $f = \ddot{q}$,

$$\frac{\partial H}{\partial P_2} = \dot{Q}_2 + P_2 \frac{\partial \dot{Q}_2}{\partial P_2} - \frac{\partial L}{\partial \ddot{q}} \frac{\partial f}{\partial P_2} = \dot{Q}_2 + P_2 \frac{\partial f}{\partial P_2} - P_2 \frac{\partial f}{\partial P_2} = \dot{Q}_2.$$

In the one for Q_1 we use the Euler-Lagrange equation,

$$\frac{\partial H}{\partial Q_1} = -\frac{\partial L}{\partial q} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}\right) = -\dot{P}_1.$$

And in the one for Q_2 we use $\dot{Q}_1 = \dot{q} = Q_2$ and $P_2 = \partial L/\partial \ddot{q}$ to obtain

$$\frac{\partial H}{\partial Q_2} = P_1 + P_2 \frac{\partial F}{\partial Q_2} - \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial \ddot{q}} \frac{\partial F}{\partial Q_2} = P_1 - \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{\partial L}{\partial \dot{q}} \tag{10}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}}\right) = -\dot{P}_2 \tag{11}$$

Thus H leads to the usual time evolution in phase space of an Hamiltonian system.

The Hamiltonian depends on P_1 linearly. Since P_1 is one of the four canonical variables, it can be set to an arbitrary initial value. Therefore H is not bounded below.

Some remarks:

- The only assumption in our derivation that the Hamiltonian is unbounded was its non-degeneracy: this means that one cannot eliminate $\ddot{q}$ in the action by partial integrations.
- Going to even higher derivatives, the situation does not improve: In general, $P_1, P_2, \ldots, P_{N-1}$ appear linearly in H, only P_N not. Thus for $N \gg 1$, half of the phase-space variables are unbounded.
- Since energy is conserved, it is classically not problematic that H is unbounded. If we go on to QFT, this property becomes however disastrous: starting from the vacuum with energy $E = 0$ we can create a two-particle state with energies $E_1 = -E_2$. Since there is a continuum of states with this energy (but different 3-momentum direction), this decay is strongly favoured w.r.t. to the inverse process (compare to the case of an excited hydrogen atom). This holds even more for the decay to $n = 3, 4, \ldots$ particles. Summing all decay rates, $n \to \infty$, the total decay rate of the vacuum diverges. Thus the simple observation that in our Universe an empty vacuum is stable excludes such systems.

See R.P. Woodard, astro-ph/0601672, for an additional discussion of the instability and the connection to gravity.