1.4 Oscillator with friction.
Consider an one-dimensional system described by the Lagrangian

$$L = e^{2\alpha t}\left[\frac{1}{2}m\dot{q}^2 - V(q)\right].$$

a.) Show that the equation of motion corresponds to an oscillator with friction term.
b). Derive the energy lost per time dE/dt of the oscillator, with $E = \frac{1}{2}m\dot{q}^2 + V(q)$.
c.) Show that the result in b.) agrees with the one obtained from the Lagrange equations of the first kind,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i$$

where the generalized forces Q^i performs the work $\delta A = Q_i \delta q^i$ (and L corresponds to the one for a conservative system).

a.) With

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = e^{2\alpha t}\left[m\ddot{q} + 2\alpha m\dot{q} + V'(q)\right] = 0$$

and $e^{2\alpha t} > 0$, the equation for an harmonic oscillator with friction term $2\alpha m\dot{q}$ follows for $V = kq^2/2$.

b.) The corresponding Hamilton function is

$$H = e^{2\alpha t}\left[\frac{1}{2}m\dot{q}^2 + V(q)\right] = e^{2\alpha t}E$$

Thus

$$\frac{dE}{dt} = -2\alpha E + e^{-2\alpha t}\frac{dH}{dt}$$

With $dH/dt = \partial H/\partial t = -\partial L/\partial t = -2\alpha L$ it follows

$$\frac{dE}{dt} = -2\alpha(E + Le^{-2\alpha t}) = -2\alpha m\dot{q}^2$$

c.) From a.) we know that $Q = -2m\alpha\dot{q}$ and thus

$$\delta A = Q\delta q = -2m\alpha\dot{q}\delta q.$$

This agrees with

$$dE = -2\alpha m\dot{q}^2 dt = -2\alpha m\dot{q}dq.$$