1.5 Driven oscillator.
Consider an harmonic oscillator satisfying

$$\ddot{q}(t)+\omega^2 q(t)=0\,,\quad \text{for}\quad t<0\quad\text{and}\quad t>T$$
$$\ddot{q}(t)-\Omega^2 q(t)=0\,,\quad \text{for}\quad 0<t<T$$

where ω and Ω are real constants. a.) Show that for $q(t)=A_1\sin(\omega t)$ for $t<0$ and $\Omega T\gg 1$, the solution

$$q(t)=A_2\sin(\omega_0 t+\alpha)$$

with $\alpha=$ const. satisfies

$$A_2\approx\frac{1}{2}(1+\omega^2/\Omega^2)^{1/2}\exp(\Omega T).$$

b.) If the oscillator was in the ground-state at $t<0$, how many quanta are created?

a.) We have to connect the solutions in the different time ranges requiring that $q(t)$ and $\dot{q}(t)$ are smooth. In the range $0<t<T$, the general solution is

$$q(t)=A_1\cosh(\Omega t)+B\sinh(\Omega t)\,.$$

Matching at $t=0$ gives $A=0$ and $B=\omega/\Omega A_1$. In the range $t>T$, the general solution is

$$q(t)=A_2\sin[\omega(t-T)+\alpha]$$

Smoothness at $t=T$ requires that $q(t)$ and $\dot{q}(t)$ match $A_2\sin(\alpha)$ and $A_2\omega\cos(\alpha)$. We eliminate the phase α using $\sin^2(\alpha)+\cos^2(\alpha)=1$,

$$\left(\frac{q(t)}{A_2}\right)^2+\left(\frac{\dot{q}(t)}{A_2\omega}\right)^2=1$$

Solving for A_2, inserting the solution for $0<t<T$ and using $\cosh^2(\alpha)=1+\sinh^2(\alpha)$ gives

$$A_2^2=A_1^2\left[1+\left(1+\frac{\omega^2}{\Omega^2}\right)\sinh^2(\Omega T)\right]$$

In the limit $\Omega T\gg 1$, we can approximate $\sinh(\Omega T)\approx\exp(\Omega T)/2\gg 1$.

b.) The energy E of a harmonic oscillator is given in general by

$$E=\frac{1}{2}(\ddot{q}+\omega^2 q^2)=\frac{1}{2}\omega^2 q^2=\left(n+\frac{1}{2}\right)\omega$$

Thus the number n of quanta is connected to the amplitude q (in the semi-classical limit) as

$$n=\frac{\omega q^2-1}{2}$$

For the ground state $n=0$ at $t<0$, the amplitude is thus $A_1=1/\sqrt{\omega}$. Setting $q=A_2$, the number n of quanta follows as

$$n=\frac{\omega A_2^2-1}{2}=\frac{1}{2}\left[\left(1+\frac{\omega^2}{\Omega^2}\right)\sinh^2(\Omega T)\right]$$