1.6 Functional derivative.
We define the derivative of a functional $F[\phi]$ by

$$\int \mathrm{d}x\, \eta(x) \frac{\delta F[\phi]}{\delta \phi(x)} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ F[\phi + \varepsilon\eta] - F[\phi] \right\}.$$

a.) Find the functional derivative of $F[\phi] = \int \mathrm{d}x \phi(x)$ and show thereby that $\delta\phi(x)/\delta\phi(x') = \delta(x - x')$. b.) Re-derive the Lagrange equations.

a.) We set $F[f] = \int_a^b \mathrm{d}x\, f(x)$ and $\eta(x) = \delta(x - x_0)$. Then

$$\frac{1}{\varepsilon} \left\{ F[f + \varepsilon\eta] - F[f] \right\} = \frac{1}{\varepsilon} \int_a^b \mathrm{d}x \left\{ f(x) + \varepsilon\delta(x - x_0) - f(x) \right\} \tag{1}$$

$$= \int_a^b \mathrm{d}x \delta(x - x_0) = \begin{cases} 1, & \text{for } x_0 \in [a,b] \\ 0, & \text{for } x_0 \notin [a,b] \end{cases} \tag{2}$$

while the LHS is zero for $x_0 \notin [a,b]$ and

$$\int_a^b \mathrm{d}x\, \delta(x - x_0) \frac{\delta F[f]}{\delta f(x)} = \begin{cases} \frac{\delta F[f]}{\delta f(x)} = \int \mathrm{d}x' \frac{\delta f(x')}{\delta f(x)}, & \text{for } x_0 \in [a,b] \\ 0, & \text{for } x_0 \notin [a,b] \end{cases}$$

Thus $\delta f(x)/\delta f(x') = \delta(x - x')$.

b.) We calculate

$$S[x + \varepsilon\eta] - S[x] = \int \mathrm{d}t \left\{ L(x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}, t) - L(x + \varepsilon\eta, x + \varepsilon\eta, t) \right\} \tag{3}$$

$$= \int \mathrm{d}t\, \varepsilon \left\{ \frac{\partial L}{\partial x} \eta + \frac{\partial L}{\partial \dot{x}} \dot{\eta} \right\} + \mathcal{O}(\varepsilon^2) \tag{4}$$

$$= \varepsilon \int \mathrm{d}t\, \left\{ \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right\} \eta + \mathcal{O}(\varepsilon^2) \tag{5}$$

where we assumed that $\eta(x)$ vanishes at the boundary points. Thus

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ S[x + \varepsilon\eta] - S[x] \right\} = \int \mathrm{d}t\, \left\{ \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}} \right\} \eta \tag{6}$$

and

$$\frac{\delta S[x]}{\delta x} = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}}. \tag{7}$$