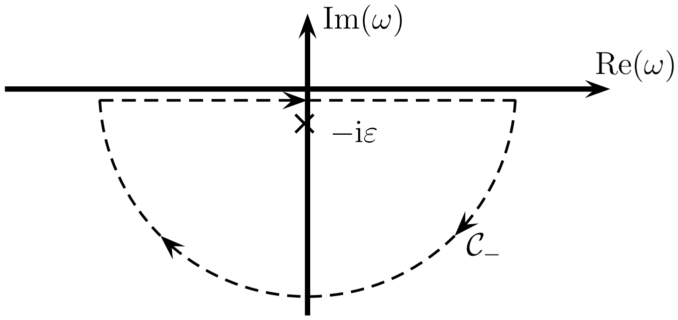
1.8 Step function.
Heaviside's step function $\vartheta(\tau)$ is defined by $\vartheta(\tau) = 0$ for $\tau < 0$ and $\vartheta(\tau) = 1$ for $\tau > 0$.
a.) Use Chauchy's residuum theorem to show that the integral representation

$$\vartheta(\tau) = -\frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega + \mathrm{i}\varepsilon}$$

is valid.
b.) Show that $\mathrm{d}\vartheta(\tau)/d\tau = \delta(\tau)$.

Using Cauchy's residue theorem, $\oint \mathrm{d}z\, f(z) = 2\pi\mathrm{i} \sum \mathrm{res}_{z_0} f(z)$, to calculate the integral requires to close the integration contour adding a path which gives a vanishing contribution to the integral. This is achieved, when the integrand $\mathrm{e}^{-\mathrm{i}\omega\tau}/\omega$ vanishes fast enough along the added path. Thus we have to choose for positive τ the contour $\mathcal{C}_-$ in the lower plane, $\mathrm{e}^{-\mathrm{i}\omega\tau} = \mathrm{e}^{-|\Im(\omega)|\tau} \to 0$ for $\Im(\omega) \to -\infty$, while we have to close the contour in the upper plane for negative τ.



a.) The residuum $\mathrm{res}_{z_0} f(z)$ of a function f with a single pole at z_0 is given by

$$\mathrm{res}_{z_0} f(z) = \lim_{z\to z_0} (z - z_0) f(z)\,. \tag{8}$$

For $\tau < 0$, no pole is inside the contour and thus $\vartheta(\tau) = 0$. For $\tau > 0$, the pole at $\omega_0 = -\mathrm{i}\varepsilon$ gives

$$\mathrm{res}_{\omega_0} f(\omega) = \lim_{\omega\to\omega_0} (\omega - \omega_0) \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega - \omega_0} = \mathrm{e}^{-\mathrm{i}\omega_0\tau}\,. \tag{9}$$

Then

$$\vartheta(\tau) = -2\pi\mathrm{i} \left(-\frac{1}{2\pi\mathrm{i}} \right) \lim_{\varepsilon\to 0} \mathrm{e}^{-\mathrm{i}(-\mathrm{i}\varepsilon)\tau} = 1\,, \tag{10}$$

where the first minus sign takes into account that the contour is oriented in the mathematical negative sense.

b.) We differentiate the integral representation,

$$\frac{\mathrm{d}\vartheta(\tau)}{\mathrm{d}\tau} = -\frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{d}\omega \frac{\mathrm{d}}{\mathrm{d}\tau} \frac{\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega + \mathrm{i}\varepsilon} \tag{11}$$

$$= -\frac{1}{2\pi\mathrm{i}} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \frac{-\mathrm{i}\omega\mathrm{e}^{-\mathrm{i}\omega\tau}}{\omega + \mathrm{i}\varepsilon} \tag{12}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\tau} = \delta(\tau) \tag{13}$$