10.5 Hyperbolic plane H^2.
The line-element of the Hyperbolic plane H^2 is given by

$$\mathrm{d}s^2 = y^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2) \qquad \text{and} \quad y \geq 0\,.$$

a. Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a{}_{bc}$. (4 pts)
b. Calculate the Riemann (or curvature) tensor $R^a{}_{bcd}$ and the scalar curvature R. (4 pts)

a. Using as Lagrange function L the kinetic energy T instead of the line-element $\mathrm{d}s$ makes calculations a bit shorter. From $L = y^{-2}(\dot{x}^2 + \dot{y}^2)$ we find

$$\frac{\partial L}{\partial x} = 0 \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{x}y^{-2}) = 2\ddot{x}y^{-2} - 4\dot{x}y^{-3}\dot{y}$$

$$\frac{\partial L}{\partial y} = -\frac{2}{y^3}(\dot{x}^2 + \dot{y}^2) \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{y}y^{-2}) = 2\ddot{y}y^{-2} - 4\dot{y}y^{-3}\dot{y}$$

and thus the solutions of the Lagrange equations are

$$\ddot{x} - 2\frac{\dot{x}\dot{y}}{y} = 0 \qquad \text{and} \qquad \ddot{y} - \frac{\dot{y}^2}{y} + \frac{\dot{x}^2}{y} = 0\,.$$

Comparing with the given geodesic equation, we read off the non-vanishing Christoffel symbols as $-\Gamma^x{}_{xy} = -\Gamma^x{}_{yx} = \Gamma^y{}_{xx} = -\Gamma^y{}_{yy} = 1/y$. (Remember that $-2y^{-1}\dot{x}\dot{y} = \Gamma^x{}_{xy}\dot{x}\dot{y} + \Gamma^x{}_{xy}\dot{x}\dot{y}$.)

b. We calculate e.g.

$$\begin{aligned} R^y{}_{xyx} &= \partial_y\Gamma^y{}_{xx} - \partial_x\Gamma^y{}_{xy} + \Gamma^y{}_{ey}\Gamma^e{}_{xx} - \Gamma^y{}_{ex}\Gamma^e{}_{xy} \\ &= -1/y^2 + 0 + \Gamma^y{}_{yy}\Gamma^y{}_{xx} - \Gamma^y{}_{xx}\Gamma^x{}_{xy} \\ &= -1/y^2 + 0 - 1/y^2 + 1/y^2 = -1/y^2\,. \end{aligned}$$

Next we remember that the number of independent components of the Riemann tensor in $d = 2$ is one, i.e. we are already done: All other components follow by the symmetry properties.
The scalar curvature is (diagonal metric with $g^{xx} = g^{yy} = y^2$)

$$R = g^{ab}R_{ab} = g^{xx}R_{xx} + g^{yy}R_{yy} = y^2(R_{xx} + R_{yy})\,.$$

Thus we have to find only the two diagonal components of the Ricci tensor $R_{ab} = R^c{}_{acb}$. With

$$\begin{aligned} R_{xx} &= R^c{}_{xcx} = R^x{}_{xxx} + R^y{}_{xyx} = 0 + R^y{}_{xyx} = -1/y^2 \\ R_{yy} &= R^c{}_{ycy} = R^x{}_{yxy} + R^x{}_{yxy} = R^x{}_{yxy} + 0 = -R^y{}_{xxy} = R^x{}_{yxy} = -1/y^2\,, \end{aligned}$$

the scalar curvature follows as $R = -2$. Hence the hyperbolic plane H^2 is a space of constant curvature, as $\mathcal{R}^2$ and S^2.
[If you wonder that $R = -2$, not -1: in $d = 2$, the Gaussian curvature K is connected to the "general" scalar curvature R via $K = R/2$. Thus $K = \pm 1$ means $R = \pm 2$ for spaces of constant unit curvature radius, S^2 and H^2. You may also check that the Riemann and Ricci tensor satisfy the relations for maximally symmetric spaces, $R_{ab} = Kg_{ab}$ and $R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})$.]