10.7 Three gauge boson vertex.
Derive the Feynman rule for the three-gluon vertex.

With $\mathrm{tr}(T^a[T^b,T^c]) = \mathrm{i}f^{bcd}\mathrm{tr}(T^aT^d) = \mathrm{i}f^{abc}/2$ we obtain

$$-\frac{1}{2}\mathrm{tr}(F^2) = -4\times\frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\mathrm{i}g)\frac{\mathrm{i}f^{abc}}{2}A^{b\mu}A^{c\nu} + \cdots = gf^{abc}\partial_\mu A^a_\nu A^{b\mu}A^{c\nu}$$

Then we go to momentum space performing a Fourier transformation,

$$\begin{aligned} F =& gf^{abc}\int \mathrm{d}^4p_1\mathrm{d}^4p_2\mathrm{d}^4p_3\,(2\pi)^4\delta(p_1+p_2+p_2)\partial_\mu A^a_\nu(p_1)A^{b\mu}(p_2)A^{c\nu}(p_3)) \\ =& -\mathrm{i}gf^{abc}p_{1\mu}\eta_{\lambda\nu}A^{a\lambda}(p_1)A^{b\mu}(p_2)A^{c\nu}(p_3) \end{aligned} \tag{208}$$

(If the vertex contains derivatives, we have to fix arbitrarily the momentum flow.) Now we can extract the vertex by taking derivatives w.r.t. to three gauge fields,

$$V^{rst}(k_1^\rho,k_2^\sigma,k_3^\tau) = \frac{\delta^3\mathrm{i}F}{\delta A^r_\rho(k_1)\,\delta A^s_\sigma(k_2)\,\delta A^t_\tau(k_3)}\,.$$

Alternatively, we can argue as follows: The antisymmetry of f^{abc} implies that the indice pairs (p_1,λ), (p_2,μ) and (p_3,ν) are antisymmetric; we make this antisymmetry explicit by the replacement

$$gf^{abc}p_{1\mu}\eta_{\lambda\nu} \to \frac{1}{3!}gf^{abc}[(p_3-p_2)_\lambda\eta_{\mu\nu} + (p_1-p_3)_\nu\eta_{\lambda\mu} + (p_2-p_1)_\mu\eta_{\nu\lambda}]$$

Finally, we note that the factor $\frac{1}{3!}$ is cancelled by the 3!possible permutations and add the factor i from $e^{\mathrm{i}S}$.