11.10 The β function in $\lambda\phi^4$.

a.) In DR, we have to add a factor $\mu^{-\varepsilon}$ to the relation between renormalised and bare coupling, $\lambda(\mu) = \mu^{-\varepsilon} Z_\phi^2 Z_\lambda^{-1} \lambda_0$. Setting $\bar{Z}^{-1} \equiv Z_\phi^2 Z_\lambda^{-1}$ and using the definition of the beta function gives

$$\beta(\lambda) = \mu \frac{\partial}{\partial \mu} \left(\mu^{-\varepsilon} \bar{Z}^{-1} \right) \lambda_0 \tag{210}$$

$$= -\varepsilon \mu^{-\varepsilon} \bar{Z}^{-1} \lambda_0 - \mu \mu^{-\varepsilon} \bar{Z}^{-2} \frac{\partial \bar{Z}}{\partial \mu} \lambda_0 \tag{211}$$

$$= -\varepsilon \lambda - \frac{\mu}{\bar{Z}} \frac{\partial \bar{Z}}{\partial \mu} \lambda \tag{212}$$

b.) Looking back at our one-loop calculations, we have $Z_\phi = 1$ and $Z_\lambda^{-1} = 1 - \frac{3\lambda}{16\pi^2\varepsilon}$. Thus $\bar{Z} = 1 + 3\lambda/(16\pi^2\varepsilon)$ and

$$\mu \frac{\partial \bar{Z}}{\partial \mu} = \frac{3}{16\pi^2\varepsilon} \mu \frac{\partial \lambda}{\partial \mu} = \frac{3}{16\pi^2\varepsilon} \beta(\lambda)\,.$$

Using $\bar{Z}^{-1} = 1 + \mathcal{O}(\lambda)$ we can write this also as

$$\frac{\mu}{\bar{Z}} \frac{\partial \bar{Z}}{\partial \mu} = \frac{3}{16\pi^2\varepsilon} \beta(\lambda)\,.$$

Inserting in (3) gives

$$\beta(\lambda) = -\varepsilon\lambda - \frac{3\lambda}{16\pi^2\varepsilon} \beta(\lambda)$$

Solving for β and taking the limit $\varepsilon \to 0$,

$$\beta(\lambda) = -\varepsilon\lambda \left(1 - \frac{3\lambda}{16\pi^2\varepsilon} \right) = \frac{3\lambda^2}{16\pi^2}\,.$$

Note that after this lengthy operation, we end up with a simple result which holds in general: The beta function is given by the coefficient of the pole in the coupling constant renormalisation constant.

c.) We set $\mathrm{d}\lambda/\mathrm{d}\ln\mu \equiv \dot{\lambda}$. Then $\dot{\lambda} = \beta(\lambda) = b_1\lambda^2 + b_2\lambda^3 + \ldots$ is the perturbative expansion of the beta function in a given scheme. The couplings in two different schemes are connected by

$$\tilde{\lambda} = \lambda + c_2\lambda^2 + \ldots \tag{213}$$

$$\lambda = \tilde{\lambda} - c_2\tilde{\lambda}^2 + \ldots \tag{214}$$

Differentiating gives

$$\dot{\lambda} = \dot{\tilde{\lambda}} - 2c_2\tilde{\lambda}\dot{\tilde{\lambda}} + \ldots = (1 - 2c_2\tilde{\lambda})\dot{\tilde{\lambda}} + \ldots$$

Solving for $\dot{\tilde{\lambda}}$, inserting first the expansion for $\dot{\lambda}$ and then for λ results in

$$\dot{\tilde{\lambda}} = (1 + 2c_2\tilde{\lambda} + \ldots)\dot{\lambda} \tag{215}$$

$$= (1 + 2c_2\tilde{\lambda} + \ldots)(b_1\lambda^2 + b_2\lambda^3 + \ldots) \tag{216}$$

$$= (1 + 2c_2\tilde{\lambda} + \ldots)[b_1(\tilde{\lambda} - c_2\tilde{\lambda}^2 + \ldots)^2 + b_2(\tilde{\lambda} - c_2\tilde{\lambda}^2 + \ldots)^3 + \ldots] \tag{217}$$

$$= b_1\tilde{\lambda}^2 + b_2\tilde{\lambda}^3 + \ldots \tag{218}$$

Thus c_2 cancels from the first two terms, and the β function is up to two loop independent from the renormalisation scheme.