## 12.1 Quantum action for a free field

Show for a free scalar field $\phi$ that the quantum action $\Gamma[\phi_c]$ coincides with the classical action $S[\phi]$.

The free generating functional for connected Green functions is

$$W_0[J] = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\, J(x)\Delta_F(x-x')J(x')\,, \tag{226}$$

and the classical field becomes

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = -\int \mathrm{d}^4x'\, \Delta_F(x-x')J(x')\,. \tag{227}$$

If we apply the Klein-Gordon operator to the classical field,

$$(\Box_x + m^2)\phi_c(x) = -\int \mathrm{d}^4x'\, (\Box_x + m^2)\Delta_F(x-x')J(x') \tag{228a}$$

$$= \int \mathrm{d}^4x'\, \delta(x-x')J(x') = J(x)\,, \tag{228b}$$

we see that $\phi_c$ is a solution of the classical field equation. Now we have all we need to write down an explicit expression for the free quantum action,

$$\Gamma_0[\phi_c] = W_0[J] - \int \mathrm{d}^4x\, J(x)\phi_c(x) \tag{229a}$$

$$= \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x'\, J(x)\Delta_F(x-x')J(x')\,. \tag{229b}$$

Inserting the expression (228) for $J(x)$ and integrating by parts twice, we obtain

$$\Gamma_0[\phi_c] = \frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^4x' \big[(\Box_x + m^2)\phi_c(x)\big]\Delta_F(x-x')\big[(\Box_{x'} + m^2)\phi_c(x')\big] \tag{230a}$$

$$= -\frac{1}{2}\int \mathrm{d}^4x\, \phi_c(x)(\Box_x + m^2)\phi_c(x) = S_0[\phi_c]\,. \tag{230b}$$

Thus the quantum action of a free field equals the classical action of the field.