**12.11 A toy model for the effective action approach.**
Consider

$$Z = \int \mathrm{d}x\mathrm{d}y \exp([-(x^2 + y^2 + \lambda x^4 + \lambda x^2 y^2)]$$

as a toy model for the generating functional of two coupled scalar fields. Integrate out the field $y$, assume then that $\lambda$ is small: Expand first the result and then rewrite it as an exponential. Show that this process results in a.) a renormalisation of the mass term $x^2$, b.) a renormalisation of the coupling term $x^4$, and c.) the appearance of new ("irrelevant") interactions $x^n$ with $n \geq 6$.

We integrate out the "heavy y", and expand then for $\lambda \ll 1$,

$$\int \mathrm{d}y \exp([-y^2(1+\lambda x^2)] = \sqrt{\frac{\pi}{1+\lambda x^2}} \simeq \sqrt{\pi}\left(1 - \frac{1}{2}\lambda x^2 + \frac{3}{8}\lambda^2 x^4 - \frac{5}{16}\lambda^3 x^6 + \cdots\right)$$

Next we want to exponentiate the $x$ terms in the bracket, but the coefficients $-1/2$, $3/8$ do not match. Comparing $(\cdots) = \exp(\sum_{n=1}^{\infty} c_{2n} x^{2n})$,

$$1 + c_2 x^2 + \left(c_4 + \frac{1}{2}c_2^2\right) x^4 + \left(c_6 + c_2 c_4 + \frac{1}{6}c_2^3\right) x^6 + \ldots = \tag{261}$$

$$1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 - \frac{5}{16}x^6 + \ldots \tag{262}$$

we find as first terms $c_2 = -1/2$, $c_4 = 1/4$, $c_6 = -1/6$, or

$$Z = \sqrt{\pi}\int \mathrm{d}x \exp\left\{-x^2\left(1+\frac{1}{2}\lambda\right) + x^4\left(\lambda - \lambda^2/4\right) - \frac{\lambda^3}{6}x^6 + \ldots\right\}$$

Note that, as final step, we should rescale the field such that the kinetic term keeps its normalisation, $x \to x/\sqrt{1+\frac{1}{2}\lambda}$.