12.2 Lambda $\Lambda_{\rm QCD}$.

a.) Show that

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1+\alpha_s(\mu^2)b_1\ln\left(Q^2/\mu^2\right)} \tag{231}$$

can be rewritten as

$$\alpha_s(Q^2) = \frac{1}{b_1\ln\left(Q^2/\Lambda^2_{\rm QCD}\right)}\,. \tag{232}$$

b.) Find the relation between $\Lambda_{\rm QCD}$ in the MS and $\overline{\rm MS}$ scheme.

a.) The general solution (in LO)

$$\frac{1}{\alpha_s(Q^2)} = \frac{1}{\alpha_s(\mu^2)} + b_1\ln\left(\frac{Q^2}{\mu^2}\right) \tag{233}$$

depends not seperately on $\alpha_s(\mu^2)$ and on μ^2, but only on a combination of these parameters. We can use instead a single parameter $\Lambda_{\rm QCD}$ defined by $\alpha_s(\Lambda_{\rm QCD})=\infty$. Integrating $\beta(\alpha_s)={\rm d}\alpha_s/{\rm d}t$ with this boundary condition gives

$$\ln(Q^2/\Lambda^2) = -\int_{\alpha_s(Q^2)}^{\infty}\frac{{\rm d}x}{\beta(x)} = -\int_{\alpha_s(Q^2)}^{\infty}\frac{{\rm d}x}{b_1x^2(1+\ldots)} = \frac{1}{b_1\alpha(Q^2)}\,, \tag{234}$$

what gives in LO the required relation $\alpha_s(Q^2)=1/(b_1\ln(Q^2/\Lambda^2))$.

b.) We integrate $\beta(\alpha_s)={\rm d}\alpha_s/{\rm d}t$, or

$$\ln(Q^2/\Lambda^2) = -\int_{\alpha_s(Q^2)}^{\infty}\frac{{\rm d}x}{\beta(x)} = -\int_{\alpha_s(Q^2)}^{\infty}\frac{{\rm d}x}{b_1x^2(1+\ldots)}\,, \tag{235}$$

in LO. We have already shown that the first two coefficients of the beta-function $\beta(\alpha_s)=b_1\alpha_s^2+b_c\alpha_s^3+\ldots$ are independent of the renormalisation scheme, $b_1=\tilde b_1$ and $b_2=\tilde b_2$. Subtracting this expression for Λ and $\tilde\Lambda$ in two different schemes gives therefore

$$\ln(\tilde\Lambda^2/\Lambda^2) = -\int_{\alpha_s(Q^2)}^{\tilde\alpha_s(Q^2)}\frac{{\rm d}x}{\beta(x)} \approx -\int_{\alpha_s(Q^2)}^{\tilde\alpha_s(Q^2)}\frac{{\rm d}x}{b_1x^2} \approx \frac{1}{2b_1}\left(\frac{1}{\alpha}-\frac{1}{\alpha(1+c_1\alpha)}\right) \approx \frac{c_1}{b_1} \tag{236}$$

for

$$\tilde\alpha_s = \alpha_s(1+c_1\alpha_s+\ldots)\,. \tag{237}$$

Thus

$$\tilde\Lambda^2 = \Lambda^2\exp(c_1/b_1). \tag{238}$$

The renormalization constant for α_s differ by $\ln(4\pi{\rm e}^{-\gamma})$ in the MS and the $\overline{\rm MS}$ scheme. Using the RGE for $\tilde\alpha_s$ and α_s we have

$$\tilde\alpha_s(Q^2) = \alpha_s(Q^2)[1+b_1\alpha_s(Q^2)\ln(4\pi{\rm e}^{-\gamma})] \tag{239}$$

or $c_1=b_1\ln(4\pi{\rm e}^{-\gamma})$. Thus

$$\Lambda_{\overline{\rm MS}} = \Lambda_{\rm MS}{\rm e}^{(\ln(4\pi)-\gamma)/2} \simeq 2.66\Lambda_{\rm MS} \tag{240}$$

(where the numerical value corresponds to $b_1(n_f=5)$), what is a sizable difference.