12.5 General solution of RGE equation – Coleman's analogue.
Find the general solution of the RGE equation for Green functions using the method of characteristics or the following analogue: Set $t = \ln \mu$, $\lambda = x$ and $\beta = v$, and associate the two derivatives with the convective derivative of a fluid containing bacteria with density $\rho = \Gamma$, and with their growth rate controlled by γ.

The RGE for renormalised 1PI Green functions has with $t = \ln \mu$ the form

$$\left[\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \lambda} - \frac{n}{2}\,\gamma\right] \Gamma_R^{(n)} = 0\,. \tag{241}$$

Equations of thus type can be solved using the method of characteristics. Alternatively, we can map the equation onto one which allows a simple interpretation: With the identifications $\beta\partial/\partial\lambda \to v\partial/\partial x$, $\Gamma_R^{(n)} \to \rho$ and $n/2\gamma \to L$, we have

$$\left[\frac{\partial}{\partial t} + v(x)\frac{\partial}{\partial x}\right] \rho(x,t) = L(x)\rho(x,t)\,. \tag{242}$$

The LHS is a convective derivative, ρ may be the density of bacteria and $L(x)$ is an arbitrary function which determines their growth rate. Aim is to find $\rho(x,t)$ for a given initial condition at $t = 0$ and prescribed $L(x)$ and $v(x)$.
We find the position $x'(t)$ of a fluid element which was at $x(t = 0)$ integrating $\mathrm{d}x'(x,t)/\mathrm{d}t = v(x')$ with initial condition $x'(x,0) = x$. Time-translation invariance implies this holds for arbitrary time differences, $x'(x, t_1 - t_2)$ and $t = t_1 - t_2$. Integrating back in time,

$$\rho(x,t) = \rho_0(x'(x,-t) \exp\left\{\int_{-t}^{0} \mathrm{d}t' L(x'(x,t'))\right\} \tag{243}$$

where $\rho_0(x)$ is an arbitrary function prescribing the initial density. Next we shift $t' \to t' + t$ and change integration variables $v(x')\mathrm{d}t' = \mathrm{d}x'(x,t)$,

$$\rho(x,t) = \rho_0(x'(x,t)) \exp\left\{\int_{0}^{t} \mathrm{d}x' \frac{L(x')}{v(x')}\right\} \tag{244}$$

Renaming variables, we obtain the desired solution to the RGE.