12.6 Vacuum polarisation and the optical theorem.
a.) Derive the imaginary part of the photon polarisation.

$$\Pi^{\rm on}(q^2)=\Pi(q^2)-\Pi(0)=\frac{2\alpha}{\pi}\int_0^1 {\rm d}x\, x(1-x)\ln\left[1-\frac{q^2}{m^2}x(1-x)\right].$$

b.) Use the optical theorem to connect $\Im(\Pi^{\rm on})$ to the decay of a virtual photon γ^* into a fermion pair, $\gamma^*\to\bar{f}f$. [Hint: Consider ${\rm d}\Phi^{(2)}\sum_{s_1,s_2}\mathcal{A}_{in}^{\mu *}\mathcal{A}_{ni}^{\nu}$ and use the tensor method.]

a.) For some background see chapter 9.1, in particular example 9.1.
We can construct the imaginary part of the photon propagator from

$$\Pi^{\rm on}(q^2)=\Pi(q^2)-\Pi(0)=\frac{2\alpha}{\pi}\int_0^1 {\rm d}x\, x(1-x)\ln\left[1-\frac{q^2}{m^2}x(1-x)\right], \tag{252}$$

using various ways:
Option 1: Find the x range for which the log is negative for a given q^2, and use then $\Im\ln(x+{\rm i}\varepsilon)=-\pi$.
From $1-\frac{q^2}{m^2}x(1-x)=0$, it follows $x_{1/2}=\frac{1}{2}\pm\frac{1}{2}\beta$ with $\beta=\sqrt{1-4m^2/q^2}$. Then

$$\Im\Pi^{\rm on}(q^2+{\rm i}\varepsilon)=\frac{2\alpha}{\pi}(-\pi)\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta}{\rm d}x\, x(1-x)=-\frac{\alpha}{3}\beta(1+2m^2/q^2)\,. \tag{253}$$

Option 2: Do first the x integral: change variables $x=\frac{1}{2}(1+\eta)$ and use then

$$\int_{-1}^{1}{\rm d}\eta\ln\left[1+x(1-\eta^2)\right]=-4(1-\vartheta\cot\vartheta) \tag{254}$$

$$\int_{-1}^{1}{\rm d}\eta\eta^2\ln\left[1+x(1-\eta^2)\right]=-\frac{4}{9}+\frac{4}{3}(1-\vartheta\cot\vartheta)\cot^2\vartheta \tag{255}$$

with $\sin^2\vartheta=-q^2/(4m^2)$. Use then ${\rm arccot}\, z={\rm i}\,{\rm arccoth}\,{\rm i}z$ to obtain the real part for $q^2>4m^2$ and

$${\rm arccoth}\, z=\frac{1}{2}\ln\frac{z+1}{1-z}$$

to find the imaginary part via the discontinuity.

b.) We apply the optical theorem to the case of a decay $1\to 1'+2'$,

$$2\Im T_{ii}=\sum_n T_{in}^*T_{ni}=\int{\rm d}\Phi^{(2)}\sum_{s_1,s_2}\mathcal{A}_{in}^*\mathcal{A}_{ni} \tag{256}$$

We want to compare the RHS to the vacuum polarisation $\Pi(q^2)$, where we factored out the external photons (and a transverse polarisation projector). Therefore we set $\mathcal{A}=\varepsilon_\mu\mathcal{A}^\mu$, and calculate with $\mathcal{A}^\mu=\bar{u}(p_1)({\rm i}e\gamma^\mu)v(p_2)$

$$\sum_{s_1,s_2}\mathcal{A}_{in}^{\mu *}\mathcal{A}_{ni}^{\nu}=e^2{\rm tr}[(\not p_1+m)\gamma^\mu(\not p_2-m)\gamma^\nu] \tag{257}$$

$$=4e^2\left[p_1^\mu p_2^\nu+p_2^\mu p_1^\nu-\frac{s}{2}\eta^{\mu\nu}\right] \tag{258}$$

where we introduced $s = k^2 = (p_1 + p_2) = 2p_1p_2 + m^2$. Then we recall

$$\mathrm{d}\Phi^{(2)} = \frac{1}{16\pi^2} \frac{|\boldsymbol{p}'_{\text{cms}}|}{\sqrt{s}} \,\mathrm{d}\Omega\,,$$

and

$$p^2_{\text{cms}} = \frac{\lambda(s, m_1^2, m_2^2)}{4s} = \frac{s}{4}\left[1 - \frac{4m^2}{s}\right]$$

for $m \equiv m_1 = m_2$. Combining this, we obtain

$$2\Im T_{ii} = \int \mathrm{d}\Phi^{(2)} \sum_{s_1,s_2} \mathcal{A}^*_{in}\mathcal{A}_{ni} = \int \mathrm{d}\Omega \frac{1}{16\pi^2} \frac{\sqrt{s}}{2\sqrt{s}} \left[1 - \frac{4m^2}{s}\right]^{1/2} 4e^2[\cdots] \tag{259a}$$

$$= 2\left[1 - \frac{4m^2}{s}\right]^{1/2} \frac{e^2}{4\pi} \int \frac{\mathrm{d}\Omega}{4\pi} [\cdots] \tag{259b}$$

$$= 2\alpha\left[1 - \frac{4m^2}{s}\right]^{1/2} S^{\mu\nu} \tag{259c}$$

with

$$S^{\mu\nu} = \int \frac{\mathrm{d}\Omega}{4\pi} \left[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \frac{s}{2}\eta^{\mu\nu}\right]. \tag{260}$$

Since $S^{\mu\nu}$ can be only a function of k, we use as ansatz

$$S^{\mu\nu} = Ak^\mu k^\nu + B\eta^{\mu\nu} \tag{261}$$

We determine the coefficients, first contracting with $k_\mu k_\nu$,

$$k_\mu k_\nu S^{\mu\nu} = s(sA + B) \tag{262}$$

$$= 2(kp_1)(kp_2) - \frac{s^2}{2} = 0 \tag{263}$$

and second contracting with $\eta_{\mu\nu}$,

$$\eta_{\mu\nu} S^{\mu\nu} = sA + 4B \tag{264}$$

$$= 2(p_1p_2) - 2s = -(s + 2m^2)^2\,. \tag{265}$$

This results in $B = -(s + 2m^2)/3$ and $A = (s + 2m^2)/3s$. Thus

$$S^{\mu\nu} = -\frac{1}{3}(1 + 2m^2/s)(k^2\eta^{\mu\nu} - k^\mu k^\nu) = S(k)(k^2\eta^{\mu\nu} - k^\mu k^\nu) \tag{266}$$

and we see that $S^{\mu\nu}$ is transverse as required by gauge invariance. Now we factor out the transverse projection operator, and consider only the scalar part of Eq. (259). Comparing then

$$\Im T_{ii} = -\frac{\alpha}{3}\left[1 - \frac{4m^2}{s}\right]^{1/2} (1 + 2m^2/s) \tag{267}$$

with a.) we find agreement.

Remark 1: We obtained $\Im\Pi(k^2)$ as follows: We cut the virtual lines, then the diagram T_{ii} decomposes into T^*_{in} and T_{ni}. The virtual fermion line corresponds now to two external on-shell particle, and therefore the integration is over $\mathrm{d}^3k/[(2\pi)^3 2\omega]$ instead $\mathrm{d}^4k/(2\pi)^4$. This shows also that the imaginary parts of loop diagrams are finite. In general, the set of such recipes to obtain the imaginary part of a loop diagram are called " Cutowsky's cutting rules".

Remark 2: From (256) we see that $\Im T_{ii} = \omega\Gamma(1 \to 1' + 2')$ holds. Thus the imaginary part $\Im T_{ii}$ should be positive. Otherwise the chosen vacuum is unstable, and the intensity of a beam of photons would grow as $I(t) = I_0 \exp(-2\Im T_{ii} t)$.