### 13.7 Effective potential in DR.

We want to evaluate

$$V_{\rm eff}^{(1)} = \frac{1}{2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\ln\left[k^2+m^2\right] \tag{261}$$

with $m^2 \equiv V'' = -\mu^2 + 3\lambda\phi^2$ in DR. In order to use our standard formulae for Feynman integrals, we should get rid of the logarithm. One possibility is to take a derivative w.r.t. $m^2$ (cf. the calculation of $\rho$ in ch. 3 or of the free energy $\mathcal{F}$ in ch. 15 of the notes). This option makes also clear that the one-loop effective potential sums up the zero-point energies. Another one is to use

$$a^\varepsilon = \mathrm{e}^{\varepsilon\ln a} = 1 + \varepsilon\ln a + \mathcal{O}(\varepsilon^2) \tag{262}$$

or

$$\frac{\partial}{\partial\varepsilon}\, a^\varepsilon|_{\varepsilon=0} = \ln(a) \tag{263}$$

Applying the second method, we should evaluate

$$= -\frac{\partial}{\partial\varepsilon}\int\frac{\mathrm{d}^dk}{(2\pi)^d}\frac{1}{(k^2+m^2)^\varepsilon}\bigg|_{\varepsilon=0} \tag{264}$$

$$= -\frac{\partial}{\partial\varepsilon}\left[\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\varepsilon-d/2)}{\Gamma(\varepsilon)}\frac{1}{(m^2)^{\varepsilon-d/2}}\right]_{\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\, f(\varepsilon)|_{\varepsilon=0}\,. \tag{265}$$

Keeping $d \neq 0, 2, 4$, the only term singular for $\varepsilon \to 0$ is $\Gamma(\varepsilon) \sim 1/\varepsilon$. With $f(\varepsilon) = \varepsilon g(\varepsilon)$ and $g(0) = \text{const.}$, it follows thus $f'(0) = g(0)$. Adding also the renormalisation scale, we arrive at

$$V_{\rm eff}^{(1)} = -\frac{1}{2}\frac{m^4}{(4\pi)^2}\,\Gamma(-d/2)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon/2}. \tag{266}$$

where now $\varepsilon \equiv 4-d$. Note the correspondence between poles of the Gamma function and power-law divergence in cutoff regularisation: $\ln(\Lambda) \leftrightarrow \Gamma(2-d/2)$, $\Lambda^2 \leftrightarrow \Gamma(1-d/2)$, and $\Lambda^4 \leftrightarrow \Gamma(-d/2)$. Including the classical potential and the counter terms, the effectice potential at one-loop is

$$V_{\rm eff}(\phi_0) = -\frac{1}{2}\mu^2\phi_0^2 + \frac{\lambda}{4}\phi_0^4 - \frac{1}{2}\frac{m^4}{(4\pi)^2}\,\Gamma(-d/2)\left(\frac{m^2}{4\pi\mu^2}\right)^{d/2} + B\phi_0^2 + C\phi_0^4 + \mathcal{O}(\hbar^2)\,. \tag{267}$$

We take the limit $\varepsilon = 4 - d \to 0$,

$$\Gamma(-d/2)\left(\frac{m^2}{4\pi\mu^2}\right)^{d/2} \to \frac{m^4}{2(4\pi)^2}\left[\frac{2}{\varepsilon} - \gamma + \ln(4\pi) - \ln(m^2/\mu^2) + \frac{3}{2}\right] \tag{268}$$

Setting $m^2 \to -\mu^2 + 3\lambda\phi_0^2$, the finite parts in $B$ and $C$ can be fixed requiring e.g.

$$\frac{\partial V_{\rm eff}}{\partial\phi}(\phi = \mu/\sqrt{\lambda}) = 0 \quad \text{and} \quad \frac{\partial^4 V_{\rm eff}}{\partial\phi^4}(\mu_0) = \lambda\,. \tag{269}$$

Or by chooseing the MS or the $\overline{\rm MS}$ schemes. As a check, you can rederive the RG equation for $\lambda(\mu)$ and $m(\mu)$ requiring the independence of the effective potential from $\mu$ (cf. for the massless case the text).