14.4 Mixing matrices.

The mixing matrices U are unitary, $UU^\dagger = 1$, and contain therefore $2n^2 - n^2$ real parameters. Because of $\omega_{ij} = -\omega_{ji}$ for a rotation, $n(n-1)/2$ parameters are angles. The remaining

$$n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$$

parameters are thus phases.
Not all phases are physical: We can reparametrise Dirac fermions, $\psi_i \to e^{i\vartheta_i}\psi_i$. These U(1) transformations keep the QED and the QCD part of the SM Langrangian invariant, since they are flavour blind. Mass terms are invariant, because we tranform L and R-fields in the same way. Thus the U(1) transformations affect only the mixing CKM matrix,

$$U \to \begin{pmatrix} e^{-i\vartheta_1} & & \\ & \cdots & \\ & & e^{-i\vartheta_n} \end{pmatrix} U \begin{pmatrix} e^{i\vartheta_{n+1}} & & \\ & \cdots & \\ & & e^{i\vartheta_{2n}} \end{pmatrix}$$

If we change all quarks by the same phase ϑ, the mixing matrix remains invariant. Thus alltogether we can eliminate only $2n-1$ phases. This leaves

$$N = \frac{n(n+1)}{2} - 2n + 1 = \frac{(n-1)(n+2)}{2}$$

physical phases: $N = 0$ for $n = 2$, and $N = 1$ for $n = 3$. Thus CP violation in the quark sector requires 3 generations.

If neutrinos are massive Dirac fermions, no difference to the quark sector arises. If they are however Majorana fermions, then there's no corresponding U(1) symmetry. We can thus eliminate only n phases, leaving

$$N = \frac{n(n+1)}{2} - n = \frac{n^2 - n}{2}$$

as physical phases, i.e. $N = 1$ for $n = 2$, and $N = 3$ for $n = 3$. Thus for $n = 3$ one can distinguish a Dirac phase and two Majorana phases. The latter two show up in physical observables only, if lepton number is violated (i.e. in $0\beta\beta$ decay, but not in neutrino oscillations).

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