15.2 Periodicity conditions.

The Matsubara Green functions are periodic on the interval $[-\beta : \beta]$. Expanding in a Fourier series, it follows

$$G(\tau) = \frac{1}{\beta} \sum_{n \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\tilde{\omega}_n \tau} G(\omega_n)$$

$$G(\omega_n) = \frac{1}{2} \int_{-\beta}^{\beta} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\tilde{\omega}_n \tau} G(\tau)$$

with $\tilde{\omega}_n = 2\pi n/(2\beta) = \pi n/\beta$. Splitting the integral into the two pieces $[-\beta : 0]$ and $0 : \beta]$, and using the periodicity condition in the first part gives

$$2G(\omega_n) = \pm \int_{-\beta}^{0} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\tilde{\omega}_n \tau} G(\tau + \beta) + \int_{0}^{\beta} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\tilde{\omega}_n \tau} G(\tau), \tag{271}$$

where the upper sign is for bosons and the lower for fermions. Next we performe the substitution $\tau \to \tau + \beta$ and use $\mathrm{e}^{\mathrm{i}\tilde{\omega}_n \beta} = \mathrm{e}^{\mathrm{i}\pi n} = (-1)^n$, obtaining

$$2G(\omega_n) = [\pm(-1)^n + 1] \int_{0}^{\beta} \mathrm{d}\tau \, \mathrm{e}^{\mathrm{i}\tilde{\omega}_n \tau} G(\tau). \tag{272}$$

For bosons, the bracket is zero for odd values of n and one if n is even, while it is opposite for fermions. Hence for bosons (fermions) only the even (odd) terms of the Fourier series contribute. Thus one defines the Matsubara frequencies as $\omega_n = 2n\pi T$ for bosons and $\omega_n = (2n + 1)\pi T$ for fermions, where again n assumes all integer values.