16.3 Scalar instantons.
Evaluating $\Box\phi$ for the ansatz gives

$$A^{-1}\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} = \frac{8\rho r^2}{(r^2+\rho^2)^3} - \frac{2\rho}{(r^2+\rho^2)^2}$$

and

$$A^{-1}\frac{3}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = -\frac{6\rho}{(r^2+\rho^2)^2}.$$

Summing the two terms, we find

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r} = 8A\rho\left[\frac{r^2}{(r^2+\rho^2)^3} - \frac{1}{(r^2+\rho^2)^2}\right] = -8A\left[\frac{\rho}{(r^2+\rho^2)}\right]^3.$$

The potential term gives

$$-\frac{\mathrm{d}V}{\mathrm{d}\phi} = -\lambda\phi^3.$$

This fixes A as $-8A-\lambda A^3=0$ or $A=\sqrt{-8/\lambda}$. Thus the coupling constant λ should be negative.

Next we calculate the action,

$$S = 2\pi^2\int_0^\infty \mathrm{d}r\, r^3\left[-\frac{1}{2}\phi\left(\frac{\mathrm{d}^2\phi}{\mathrm{d}r^2} + \frac{3}{r}\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) + V(\phi)\right]$$

With

$$\rho^4\int_0^\infty \mathrm{d}r\,\frac{r^3}{(r^2+\rho^2)^4} = \frac{1}{12}$$

we find

$$S = \frac{\pi^2}{6}\left[4A^2 + \frac{\lambda}{4}A^4\right] = \frac{\pi^2}{6}\left[4(-8/\lambda) + \frac{\lambda}{4}(-8/\lambda)^2\right] = \frac{8\pi^2}{3\lambda}.$$

For background and generalisation to arbitrary spacetime dimension d see S. Fubini, "A New Approach to Conformal Invariant Field Theories," Nuovo Cim. A **34**, 521 (1976).

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