18.10 Expansion of the Γ function.

We start from the recurrence relation for the digamma function,

$$\psi(z+1) = \frac{1}{z} + \psi(z)\,, \tag{283}$$

which becomes for the special case of non-negative integers $n = 0, 1, 2, \ldots$

$$\psi(n+1) = \frac{1}{n} + \psi_1(n) = \frac{1}{n} + \frac{1}{n-1} + \ldots + \psi(1)\,. \tag{284}$$

In order to differentiate ψ, we generalize this relation to arbitrary z,

$$\psi(z+n) = \frac{1}{z+n-1} + \frac{1}{z+n-2} + \ldots + \frac{1}{z} + \psi(z) \tag{285}$$

$$= \sum_{k=0}^{n-1} \frac{1}{z+k} + \psi(z)\,. \tag{286}$$

Subtracting (286) from (284), we obtain

$$\psi(z+n) - \psi(z+1) = \sum_{k=0}^{n-1} \left(\frac{1}{z+k} - \frac{1}{k+1} \right) + \psi(z) + \gamma\,. \tag{287}$$

Taking the limit $n \to \infty$, it follows (see also appendix)

$$\lim \psi(z+n) - \psi(z+1) = \mathcal{O}(1/n) \to 0 \tag{288}$$

and thus

$$\psi(z) = -\gamma - \sum_{k=0}^{\infty} \left(\frac{1}{z+k} - \frac{1}{k+1} \right) \tag{289}$$

Now we can calculate the second logarithmic derivative of the Γ function,

$$\frac{\mathrm{d}^2 \ln \Gamma(z)}{\mathrm{d}z^2} = \frac{\mathrm{d}\psi(z)}{\mathrm{d}z} = \sum_{k=0}^{\infty} \left(\frac{1}{z+k} \right)^2 \tag{290}$$

Next we use

$$\frac{\mathrm{d}^2 \ln \Gamma(z)}{\mathrm{d}z^2} = \frac{\mathrm{d}}{\mathrm{d}z} \frac{\Gamma'(z)}{\Gamma(z)} = \frac{\Gamma''(z)}{\Gamma(z)} - \left[\frac{\Gamma'(z)}{\Gamma(z)} \right]^2\,, \tag{291}$$

set $z = 1$, insert $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma$ and obtain

$$\Gamma''(1) - \gamma^2 = \sum_{k=0}^{\infty} \left(\frac{1}{1+k} \right)^2 = \frac{\pi^2}{6}\,. \tag{292}$$

Thus the expansion of $\Gamma(1+\varepsilon)$ is

$$\Gamma(1+\varepsilon) = 1 - \gamma\varepsilon + \frac{1}{2} \left(\gamma^2 + \frac{\pi^2}{6} \right) \varepsilon^2 + \mathcal{O}(\varepsilon^3)\,. \tag{293}$$

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