### 19.7 Riemann tensor.

Inserting the definition of the Christoffel symbols and using normal coordinates (and for simplicity Latin indices), the Riemann tensor becomes

$$R_{abcd} = \frac{1}{2}\{\partial_d\partial_b g_{ac} + \partial_c\partial_a g_{bd} - \partial_d\partial_a g_{bc} - \partial_c\partial_b g_{ad}\}\,. \tag{307}$$

The tensor is antisymmetric in the indices $c \leftrightarrow d$, antisymmetric in $a \leftrightarrow b$ and symmetric against an exchange of the index pairs $(ab) \leftrightarrow (cd)$. Moreover, there exists one algebraic identity,

$$R_{abcd} + R_{adbc} + R_{acdb} = 0\,, \tag{308}$$

which is shown by inserting (307). Since each pair of indices $(ab)$ and $(cd)$ can take six values, we can combine the antisymmetrized components of $R_{[ab][cd]}$ in a symmetric six-dimensional matrix. The number of independent components of this matrix is thus for $d = 4$ space-time dimensions

$$\frac{n \times (n+1)}{2} - 1 = \frac{6 \times 7}{2} - 1 = 20\,,$$

where we accounted also for the constraint (308). In general, the number $n$ of independent components is in $d$ space-time dimensions given by $n = d^2(d^2-1)/12$, while the number $m$ of field equations is $m = d(d+1)/2$. Thus we find

| $d$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $n$ | 0 | 1 | 6 | 20 |
| $m$ | - | 3 | 6 | 10 |

This implies that an one-dimensional manifold is always flat (ask yourself for a simple proof). Moreover, the number of independent components of the Riemann tensor is smaller or equals the number of field equations for $d = 2$ and $d = 3$. Hence the Riemann tensor vanishes in empty space, if $d = 2, 3$. Starting from $d = 4$, already an empty space can be curved. This corresponds to our earlier result in problem 7.4, that gravitational waves exist only in $d \geq 4$.

The Bianchi identity is a differential constraint (corresponding to $\partial_\mu \tilde{F}^{\mu\nu} = 0$ in the YM case),

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0\,, \tag{309}$$

that is checked again simplest using normal coordinates. In the context of general relativity, the Bianchi identities are necessary consequence of the Einstein-Hilbert action and the requirement of general covariance.