2.1 Classical action.
Calculate the classical action $S[q]$ for a free particle and an harmonic oscillator. Compare the results with the expression for the propagator $K = \langle x', t'|x, t\rangle = N\exp(\mathrm{i}\phi)$ of the corresponding quantum mechanical system and express both ϕ and N through the action S.

a. We use the non-relativistic expression $L = m\dot{x}^2/2$ and express the velocity of a particle following a classical path between a and b as

$$\dot{x} = \frac{x_b - x_a}{t_b - t_a}$$

Hence

$$S = \int_a^b \mathrm{d}tL = \frac{m}{2}\frac{(x_b - x_a)^2}{t_b - t_a}\,.$$

b.) Now $L = m\dot{x}^2/2 - m\omega^2x^2/2$ and $\ddot{x} + \omega^2 x = 0$. From

$$x(t) = A\cos\omega(t - t_a) + B\sin\omega(t - t_a)$$

we find $A = x_a$ and $B = (x_b - x_a\cos\omega(t - t_a))/\sin\omega(t - t_a)$. Hence

$$S = \frac{m}{2}\int_a^b \mathrm{d}t(\dot{x}^2 - \omega^2x^2) = \frac{m\omega}{4}\left\{(B^2 - A^2)\sin(2\omega(t_b - t_a)) + 2AB(\cos(2\omega(t_b - t_a) - 1)\right\}$$

Inserting A and B gives

$$S = \frac{m\omega}{2\sin(2\omega(t_b - t_a))}\left\{(x_a^2 + x_b^2)\cos(\omega(t_b - t_a)) - 2x_ax_b\right\}$$

Comparing our results we the expression for the propagator $K = N\exp(\mathrm{i}\phi)$ of the corresponding quantum mechanical system, we see that its phase is given by the classical action.