2.10 Statistical mechanics.
Derive the connection between the partition function of statistical mechanics and the path integral in quantum mechanics,

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \sum_n \langle n| \, \mathrm{e}^{-\beta H} \, |n\rangle \Leftrightarrow \int \mathcal{D}q(t) \exp\left(\mathrm{i} \int \mathrm{d}t L\right) .$$

considering the latter in Euclidean time $\tau = t' - t = -\mathrm{i}\beta$.

Recall the definition of the propagator and its connection to the path integral,

$$K(q', t'; q, t) = \left\langle q' \right| \mathrm{e}^{-\mathrm{i}H(t'-t)} \left| q \right\rangle = \int_{q(t)=q_i}^{q(t')=q_f} \mathcal{D}q \, \mathrm{e}^{\mathrm{i}S} .$$

Consider now the effect of a Wick rotation, i.e. rotate the time axis clockwise by 90 degrees in the complex plane. This corresponds to the transition from Minkowski to Euclidean space,

$$x^2 = t^2 - \boldsymbol{x}^2 \to x_E^2 = -[t_E^2 + \boldsymbol{x}^2] .$$

Performing the changes $t_E = \mathrm{i}t$, $\mathrm{d}t = -\mathrm{i}t_E$, and $\partial_t = \mathrm{i}\partial t_E$ in the action of a particle moving in an one-dimensional potential gives

$$S = -\mathrm{i} \int \mathrm{d}t_E \left(-\frac{1}{2} m \dot{q}_E^2 - V(q) \right) \equiv \mathrm{i} S_E . \tag{40}$$

Note that the Euclidean action $S_E = T_E + V = \int \mathrm{d}tH$ is bounded from below. The phase factor in the path-integral transforms as $\mathrm{e}^{\mathrm{i}S} = \mathrm{e}^{-S_E}$, and thus the contribution from large S_E is exponentially damped in the Euclidean path-integral.
Setting $\tau = t' - t = -\mathrm{i}\beta$, $q' = q$ and integrating over all periodic paths q, we get

$$\sum_q \langle q| \, \mathrm{e}^{-\beta H} \, |q\rangle = \mathrm{Tr}\, \mathrm{e}^{-\beta H} = \int_{q(t)=q(t+\beta)} \mathcal{D}q \, \mathrm{e}^{-S_E} .$$